

Analysis of Thin Non-Newtonian Fluid Film on a Wall using OHAM and HPM

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ABSTRACT

This work investigates the Johnson-Segalman fluid's magneto-hydrodynamic (MHD) thin-film flow in lifting conditions on a vertical surface. The basic equations of linear momentum and continuity that govern the flow are reduced to an ordinary differential equation. To solve the nonlinear differential equation that arises from the mathematical modelling of the flow, we have used the Optimal Homotopy Asymptotic Method and Homotopy Perturbation Method. The effect of various non-dimensional parameters, such as the applied magnetic field, Weissenberg number, and Stokes number, on the velocity profile is elaborated through various graphs.

Keywords: Non-Newtonian fluid, thin film flow, MHD, OHAM, HPM

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1 | INTRODUCTION

Non-Newtonian fluids are used on a large scale in industrial and technological methods, and they are gaining importance in scientific and research circles. Because of their many advantages, non-Newtonian fluids are utilized in many engineering, industrial, and other high-tech fields. Paints, toothpaste, shampoo, plastics, biological products, and various adhesive products are among the industries that primarily use this field. The nonlinear relationship between the stress tensor and the deformation rate essentially sets these fluids apart. Many fluids can be defined using different models. One of the many important fluid modes that are widely used in manufacturing and engineering is Johnson-Segalman fluid.

2 | LITERATURE REVIEW

Different characteristics are displayed by non-Newtonian Johnson-Segalman thin-film flow, which was initially studied by [1]. Various fluid models could be used to explain the non-Newtonian behavior of fluids. Many scholars are interested in the Johnson-Segalman fluid because it represents the Maxwell and Oldroyd B fluids, two examples of classical Newtonian fluids [2]. Understanding the dynamics of Johnson-Segalman fluid flow between two revolving cylinders, with a focus on suction, is the main goal [3]. Understanding the properties of non-Newtonian fluids in rotating systems is crucial for many processes, such as combining, pumping, and extrusion. [4] used Johnson-Segalman fluids to study three distinct cylinder Poiseuille flows. This research could be useful for any field that deals with non-Newtonian fluids, such as biological fluids, suspensions, and polymers. The Johnson-Segalman fluid is attracting the attention of more researchers. [5] and [6] studied the aforementioned fluid flow on surfaces that move vertically. [7] investigated the fractional Johnson-Segalman fluid model when thin film flows along a vertical strip. [8] studied the steady state thin film flow of Johnson-Segalman fluid on vertical cylinders under lifting and draining circumstances. The non-Newtonian MHD fluid's behavior when a thin layer passes across a moving vertical belt is examined by [9].

The MHD thin-film flow of Johnson-Segalman fluids on raised surfaces under lift conditions is studied in the current work. For non-affine distortion, the Johnson-Segalman viscoelastic fluid model was created [10]. Many researchers have used this non-Newtonian paradigm to examine and interpret the spurt phenomenon [11], [12]. Research has been conducted on the issue, and researchers frequently associate a spurt with a slide on the wall [13], [14]. This model was also used for peristaltic flow by [15]. For the solution of integral equations, nonlinear and linear ordinary (partial differential equations) that occur in applied science and engineering, many investigators have employed the Adomian decomposition approach [16], [17], [18], [19]. In our instance, we additionally solve MHD thin-film flow using the OHAM and HPM, which results in an infinite series result with easily known

components. The OHAM and HPM offer an approximate solution to the nonlinear problem without any simplification, in contrast to common approaches in previous literature, such as weak nonlinearity and minor perturbation, which alter the physics of the problem as a result of such simplification [20], [21].

This study makes a significant contribution to the field of non-Newtonian fluid dynamics by investigating the Johnson-Segalman fluid's magneto-hydrodynamic (MHD) thin-film flow on a vertical surface. The novelty of this research lies in its application to OHAM and HPM methods for the solution of nonlinear differential equations arising from the Johnson-Segalman fluid model, providing new insights into the behaviour of non-Newtonian fluids in complex systems.

The structure of this paper is as follows. The literature review is given in Section 2. The governing equations of the Johnson-Segalman model are described in Section 3. The lift problem and flow equation are covered in Section 4. The fundamental ideas of the OHAM and HPM, together with the problem's solution, will be covered in Section 5. In Section 6, the problem's solution is covered. Section 7 provides the average velocity and volume flow rate. Section 8 presents our findings and discussion. Section 9 presents the conclusion.

3 | FUNDAMENTAL EQUATIONS

The mathematical equations governing a non-compressible Johnson-Segalman fluid's MHD flow are

$$\operatorname{div} \mathbf{V} = 0, \quad (1)$$

$$\rho \frac{D\mathbf{V}}{Dt} = \operatorname{div} \mathbf{T} + \rho \mathbf{f} + \mathbf{J} \times \mathbf{B}, \quad (2)$$

where $\mathbf{V} = [u(x, y, z), v(x, y, z), w(x, y, z)]$ is the fluid's velocity vector, $\frac{D}{Dt}$ is the material time derivative, $\mathbf{f}$ is the body force per unit mass, and ρ denotes the constant density of the fluid. The formula for the force of Lorentz per unit volume is

$$\mathbf{J} \times \mathbf{B} = [0, -\sigma B_0^2 v(x), 0], \quad (3)$$

$$\mathbf{T} = -p\mathbf{I} + \mathbf{S} + 2\mu\mathbf{D}, \quad (4)$$

$$\mathbf{S} + m \left[\frac{D\mathbf{S}}{Dt} + \mathbf{S}(\mathbf{W} - a\mathbf{D}) + (\mathbf{W} - a\mathbf{D})^T \mathbf{S} \right] = 2\eta\mathbf{D}, \quad (5)$$

The symbols for current density are $\mathbf{J}$, the magnetic field $\mathbf{B}$, pressure p , identity tensor $\mathbf{I}$, extra stress tensor $\mathbf{S}$, viscosities μ and η , relaxation time m , and slip parameter a are used in equations (4) and (5). $\mathbf{W}$ stands to represent the skew-symmetric part of the velocity gradient, whereas $\mathbf{D}$ represents the symmetric part, and they are defined as

$$\mathbf{W} = \frac{1}{2}[\mathbf{L}^T - \mathbf{L}], \mathbf{D} = \frac{1}{2}[\mathbf{L}^T + \mathbf{L}] \text{ and } \mathbf{L} = \operatorname{grad} \mathbf{V}. \quad (6)$$

4 | PROBLEM STATEMENT

A thin layer of a non-compressible and non-Newtonian MHD Johnson-Segalman fluid flow [10] is considered in this work. It is assumed that a wide belt is moving in the upward direction with a constant speed. U_0 as shown in Figure 1. Fluid drains down the belt due to gravity in the form of a thin film of uniform thickness δ . Taking y -axis along the belt and x axis normal to the belt. The following boundary conditions apply to the problem:

$$T_{xy} = 0 \quad \text{At } x = \delta, \text{ (free surface)} \quad (7)$$

$$v(x) = U_0, \quad \text{At } x = 0, \text{ (no slip condition),} \quad (8)$$

T_{xy} Represents the Johnson-Segalman fluid's shear stress component. We consider the velocity profile and the stress tensor in the form.

$$\mathbf{V} = (0, v(x), 0), \quad T = T(x) \quad (9)$$

After simplifying equation (2), the following equations [22] are obtained

$$\begin{aligned} \frac{d^2v}{dx^2} + \phi W_e^2(1-a^2) \left[\phi W_e^2(1-a^2) \left(\frac{dv}{dx} \right)^4 \frac{d^2v}{dx^2} + (3\phi-1) \left(\frac{dv}{dx} \right)^2 \frac{d^2v}{dx^2} \right] - W_e^4(1-a^2)^2 \left[S_t \left(\frac{dv}{dx} \right)^4 + \right. \\ \left. M^2 v(x) \left(\frac{dv}{dx} \right)^4 \right] - 2W_e^2(1-a^2) \left[S_t \left(\frac{dv}{dx} \right)^2 + M^2 v(x) \left(\frac{dv}{dx} \right)^2 \right] - M^2 v(x) = S_t, \end{aligned} \quad (10)$$

where $\phi = \frac{\mu}{\mu+\eta}$ represents the ratios of viscosities, $W_e = \frac{mU_0}{\delta}$, the Weissenberg number (A Weissenberg number is the ratio of the relaxation time of the fluid to a characteristic time associated with the flow.) $S_t = \frac{\rho\delta^2g}{\mu_{eff}U_0}$, the Stokes number, $M^2 = \frac{\sigma\delta(B_0)^2}{\mu_{eff}}$ and $\mu_{eff} = \mu + \eta$.

The following boundary conditions apply to the problem:

$$\frac{dv}{dx} = 0 \quad \text{at } x = 0, \quad (11)$$

$$v = 1 \quad \text{at } x = 0. \quad (12)$$

The Homotopy perturbation technique (HPM) and the Optimal Homotopy Asymptotic method (OHAM) are currently used to solve the resulting ordinary differential equation with associated boundaries.

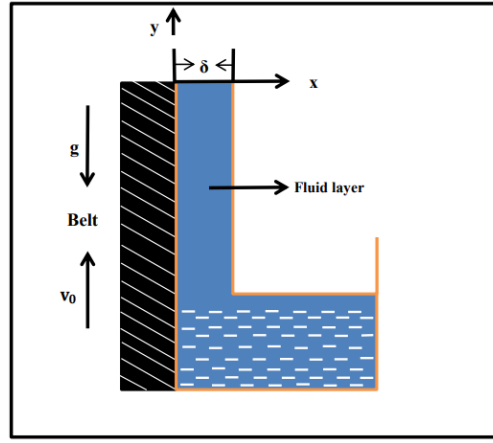


Figure 3.1: Geometry of the problem

5 | FUNDAMENTAL IDEAS OF METHODS

5.1. Fundamentals of Optimal Homotopy Asymptotic Method (OHAM)

Consider a non-linear differential equation.

$$L(\tilde{V}(\tilde{\delta})) + g(\tilde{\delta}) + N(\tilde{V}(\tilde{\delta})) = 0, \quad B\left(\tilde{V}, \frac{d\tilde{V}}{d\tilde{\delta}}\right) = 0. \quad (13)$$

The non-linear operator is denoted by $N(\tilde{V}(\tilde{\delta}))$, the source term is $g(\tilde{\delta})$, the linear operator is L , the unknown function is $\tilde{V}(\tilde{\delta})$, and the boundary condition is B .

Using the Optimal Homotopy Asymptotic Method (OHAM), we were able to obtain the following.

$$(1-r)[L(\tilde{V}(\tilde{\delta}, r)) + g(\tilde{\delta})] = H(r)[L(\tilde{V}(\tilde{\delta}, r)) + g(\tilde{\delta}) + N(\tilde{V}(\tilde{\delta}, r))], \quad B\left(\tilde{V}(\tilde{\delta}, r), \frac{d\tilde{V}(\tilde{\delta}, r)}{d\tilde{\delta}}\right) = 0. \quad (14)$$

The secondary function of H understand that it is non-zero for $r \neq 0$ and that it is definitely given for $r = 0$ (i.e., $H(0) = 0$) for $r = 0$ and $r = 1$. The expression for the inserting variable $r \in [0, 1]$ is

$$\tilde{V}(\tilde{\delta}, 0) = \tilde{V}_0(\tilde{\delta}), \quad \tilde{V}(\tilde{\delta}, 1) = \tilde{V}(\tilde{\delta}). \quad (15)$$

With r ranging from 0 to 1, the solution $\tilde{V}(\tilde{\delta}, r)$ fluctuates between $\tilde{V}_0(\tilde{\delta})$ and $\tilde{V}(\tilde{\delta})$. By adjusting $r = 0$ in Eq. (13) we obtain $\tilde{V}_0(\tilde{\delta})$.

$$L(\tilde{V}_0(\tilde{\delta})) + g(\tilde{\delta}) = 0, \quad B\left(\tilde{V}_0, \frac{d\tilde{V}_0}{d\tilde{\delta}}\right) = 0. \quad (16)$$

The auxiliary function, $H(r)$, can be written as

$$H(r) = rc_1 + r^2c_2 + r^3c_3 + \dots \quad (17)$$

To determine the constants in this case, $c_1, c_2, c_3, \dots$, Eq. (14) may be stated as

$$\tilde{V}(\tilde{\delta}, r, c_i) = \tilde{V}_0(\tilde{\delta}) + \sum_{j \geq 1} \tilde{V}_j(\tilde{\delta}, c_i) r^j, \quad i = 1, 2, \dots \quad (18)$$

After Comparable powers of r are analyzed, and Eq. (18) is substituted in Eq. (14), yielding the system below.

$$L(\tilde{V}_1(\tilde{\delta})) = c_1 N_0(\tilde{V}_0(\tilde{\delta})), \quad B\left(\tilde{V}_1, \frac{d\tilde{V}_1}{d\tilde{\delta}}\right) = 0. \quad (19)$$

$$L(\tilde{V}_j(\tilde{\delta}) - \tilde{V}_{j-1}(\tilde{\delta})) = c_j N_0(\tilde{V}_0(\tilde{\delta})) + \sum_{k=1}^{j-1} c_k [L(\tilde{V}_{j-k}(\tilde{\delta})) N_{j-k}(\tilde{V}_0(\tilde{\delta}), \tilde{V}_1(\tilde{\delta}), \dots, \tilde{V}_{j-1}(\tilde{\delta}))], \quad B\left(\tilde{V}_1, \frac{d\tilde{V}}{d\tilde{\delta}}\right) = 0, \quad j = 1, 2, \dots \quad (20)$$

In Eq. (20) the term $N_m(\tilde{V}_0(\tilde{\delta}), \tilde{V}_1(\tilde{\delta}), \dots, \tilde{V}_n(\tilde{\delta}))$ is the coefficient of r^m in the expansion of

$$N(\tilde{V}(\tilde{\delta}, r, c_k)) = N_0(\tilde{V}_0(\tilde{\delta})) + \sum_{j \geq 1} N_j \tilde{V}_0(\tilde{\delta}), \tilde{V}_1(\tilde{\delta}), \dots, \tilde{V}_j(\tilde{\delta}) r^j, \quad k = 1, 2, \dots \quad (21)$$

The results of Eq. (18) can be simply computed for $\tilde{V}_j(\tilde{\delta}), j \geq 0$, meaning that the meeting is completely dependent on the constants $c_1, c_2, c_3, \dots$. If it is convergent at $r = 1$, then Eq. (18) requires

$$\tilde{V}(\tilde{\delta}, c_k) = \tilde{V}_0(\tilde{\delta}) + \sum_{j \geq 1} \tilde{V}_j(\tilde{\delta}, c_i). \quad (22)$$

Equation (13)'s solution is typically represented as

$$\tilde{V}^n(\tilde{\delta}, c_k) = \tilde{V}_0(\tilde{\delta}) + \sum_{j=1}^n \tilde{V}_j(\tilde{\delta}, c_k), \quad k = 1, 2, \dots, n. \quad (23)$$

The residual is determined by inserting Equation (16) into Equation (23).

$$R(\tilde{\delta}, c_k) = L(\tilde{V}^n(\tilde{\delta}, c_k)) + g(\tilde{\delta}) + N(\tilde{V}^n(\tilde{\delta}, c_k)), \quad k = 1, 2, \dots, n. \quad (24)$$

When $R(\tilde{\delta}, c_k) = 0$, i.e., At zero residual, we obtain the precise answer $\tilde{V}^n(\tilde{\delta}, c_k)$, then if $R(\tilde{\delta}, c_k) \neq 0$, that is, if the residual is greater than zero, we can reduce in this manner.

$$J(c_l) = \int_a^b R^2(\omega, c_l) dx. \quad (25)$$

With a and b being constants depending on the problem at together, the conditions can be utilized to compute the unknown constants. $c_1, c_2, c_3, \dots$, and so on,

$$\frac{\partial J}{\partial c_l} = 0, \quad l = 1, 2, \dots, n/. \quad (26)$$

Equation (16) can be used to determine the approximated solution once we have determined the values of these constants. Furthermore, we apply the collocation strategy to estimate the value of the unidentified constant: $c_1, c_2, c_3, \dots$

5.2. Fundamentals of Homotopy Perturbation Method (HPM)

The mathematical structure of HPM may be ascertained by applying boundary conditions to the non-linear differential equation.

$$C(j) = g(n); n \in \Omega, K\left(j, \frac{\partial j}{\partial \delta}\right), n \in I, \quad (27)$$

I represents the domain border, K represents the boundary operator, $C(j)$ is the differential operator, and $g(n)$ is an analytical function. Suppose that Equation (27) may be rewritten as follows.

$$H(j) + N(j) = g(n). \quad (28)$$

Using a Homotopy

$$\alpha(n, h): \Omega \times [0, 1] \rightarrow M, \quad (29)$$

That fulfill

$$K\alpha(n, h) = (1 - h)[H(v) - H(w_0) + h[C(\alpha) - g(n)] = 0, \quad (30)$$

where $h \in [0,1]$ is an insertion operator, which is regarded as a tiny parameter, and w_0 is the main estimate of equation (27). The homology method can be used to derive the series solution of equation (30) as follows.

$$v = v_0 + hv_1 + h^2v_2 + \dots, \quad (31)$$

If $h \rightarrow 1$ equation (31) is used to obtain the approximate solution to Equation(30), then

$$j(x) = \lim_{h \rightarrow 1} v = v_0 + hv_1 + h^2v_2 + \dots, \quad (32)$$

6 | SOLUTIONS OF THE PROBLEM

6.1. Solutions of the Problems Using OHAM

Using OHAM, the various component problems and their associated boundary conditions are given below.

6.1.1. Zeroth Component Problems

The zeroth component problem of equation (10), as well as its associated boundary conditions, are

$$\left(\frac{d^2v_0}{dx^2}\right) - S_t = 0, \quad (33)$$

$$v_0(0) = 1, \quad v_0'(1) = 0. \quad (34)$$

6.1.2. First Component Problems

The first component problem of equation (10), as well as its associated boundary conditions, are

$$\begin{aligned} & \left(\frac{d^2v_1}{dx^2}\right) - a^4\phi^2C_1W_e^4\left(\frac{dv_0}{dx}\right)^4\left(\frac{d^2v_0}{dx^2}\right) + 2a^2\phi^2C_1W_e^4\left(\frac{dv_0}{dx}\right)^4\left(\frac{d^2v_0}{dx^2}\right) - \phi^2C_1W_e^4\left(\frac{dv_0}{dx}\right)^4\left(\frac{d^2v_0}{dx^2}\right) + \\ & 3a^2\phi^2C_1W_e^2\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right) - 3\phi^2C_1W_e^2\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right) - a^2\phi C_1W_e^2\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right) + \phi C_1W_e^2\left(\frac{dv_0}{dx}\right)^2\left(\frac{d^2v_0}{dx^2}\right) - \\ & C_1\left(\frac{d^2v_0}{dx^2}\right) - \left(\frac{d^2v_0}{dx^2}\right) + a^4M^2C_1W_e^4v_0(x)\left(\frac{dv_0}{dx}\right)^4 - 2a^2M^2C_1W_e^4v_0(x)\left(\frac{dv_0}{dx}\right)^4 + M^2C_1W_e^4v_0(x)\left(\frac{dv_0}{dx}\right)^4 + \\ & a^4C_1S_tW_e^4\left(\frac{dv_0}{dx}\right)^4 - 2a^2C_1S_tW_e^4\left(\frac{dv_0}{dx}\right)^4 + C_1S_tW_e^4\left(\frac{dv_0}{dx}\right)^4 - 2a^2M^2C_1W_e^2v_0(x)\left(\frac{dv_0}{dx}\right)^2 + \\ & 2M^2C_1W_e^2v_0(x)\left(\frac{dv_0}{dx}\right)^2 - 2a^2C_1S_tW_e^2\left(\frac{dv_0}{dx}\right)^2 + 2C_1S_tW_e^2\left(\frac{dv_0}{dx}\right)^2 + M^2C_1v_0(x) + C_1S_t + S_t = 0 \end{aligned} \quad (35)$$

$$v_1(0) = 0, \quad v_1'(1) = 0. \quad (36)$$

6.2. Solutions of the Problem Using HPM

Using HPM, the various component problems and their associated boundary conditions are given below.

6.2.1. Zeroth Component Problems

The zeroth component problem, as well as the associated boundary conditions, are

$$\left(\frac{d^2 v_0}{dx^2}\right) - S_t = 0 \quad (37)$$

$$v_0(0) = 1, \quad v_0'(1) = 0. \quad (38)$$

6.2.2. First Component Problems

The first component problem, as well as its associated boundary conditions, are

$$\begin{aligned} & \frac{d^2 v_1}{dx^2} + a^4 \phi^2 W_e^4 \left(\frac{dv_0}{dx}\right)^4 \left(\frac{d^2 v_0}{dx^2}\right) - 2a^2 \phi^2 W_e^4 \left(\frac{dv_0}{dx}\right)^4 \left(\frac{d^2 v_0}{dx^2}\right) + \phi^2 W_e^4 \left(\frac{dv_0}{dx}\right)^4 \left(\frac{d^2 v_0}{dx^2}\right) - \\ & 3a^2 \phi^2 W_e^2 \left(\frac{dv_0}{dx}\right)^2 \left(\frac{d^2 v_0}{dx^2}\right) + 3\phi^2 W_e^2 \left(\frac{dv_0}{dx}\right)^2 \left(\frac{d^2 v_0}{dx^2}\right) + a^2 \phi W_e^2 \left(\frac{dv_0}{dx}\right)^2 \left(\frac{d^2 v_0}{dx^2}\right) - \phi W_e^2 \left(\frac{dv_0}{dx}\right)^2 \left(\frac{d^2 v_0}{dx^2}\right) - \\ & a^4 M^2 W_e^4 v_0(x) \left(\frac{dv_0}{dx}\right)^4 + 2a^2 M^2 W_e^4 v_0(x) \left(\frac{dv_0}{dx}\right)^4 - M^2 W_e^4 v_0(x) \left(\frac{dv_0}{dx}\right)^4 - a^4 S_t W_e^4 \left(\frac{dv_0}{dx}\right)^4 + \\ & 2a^2 S_t W_e^4 \left(\frac{dv_0}{dx}\right)^4 - S_t W_e^4 \left(\frac{dv_0}{dx}\right)^4 + 2a^2 M^2 W_e^2 v_0(x) \left(\frac{dv_0}{dx}\right)^2 - 2M^2 W_e^2 v_0(x) \left(\frac{dv_0}{dx}\right)^2 + \\ & 2a^2 S_t W_e^2 \left(\frac{dv_0}{dx}\right)^2 - 2S_t W_e^2 \left(\frac{dv_0}{dx}\right)^2 - M^2 v_0(x) = 0 \end{aligned} \quad (39)$$

$$v_1(0) = 0, \quad v_1'(1) = 0. \quad (40)$$

Table 1. Velocity profile by using the OHAM method, keeping $W_e = 0.001, \phi = 1.3$,
 $a = 0.5, M = 0.007, S_t = 1.7$.

x	v_{OHAM}	Residual v_{OHAM}
0	1	$-4.548674951916675 \times 10^{-11}$
0.1	0.838498295058262	$-2.902051528637849 \times 10^{-11}$
0.2	0.693996948735602	$-2.383279663030479 \times 10^{-16}$
0.3	0.5664959013147015	$3.681183804573403 \times 10^{-11}$
0.4	0.455995100103893	$7.721360497256968 \times 10^{-11}$
0.5	0.36249449943712864	$1.175628508862862 \times 10^{-10}$
0.6	0.2859940606739523	$1.547776925106081 \times 10^{-10}$
0.7	0.22649375219947687	$1.863374519364447 \times 10^{-10}$
0.8	0.1839935494243653	$2.102808711628345 \times 10^{-10}$
0.9	0.15849343478481603	$2.252072152591035 \times 10^{-10}$
1	0.1499933977425525	$2.302758232298798 \times 10^{-10}$

Table 2. Velocity profile by using the HPM method, keeping $W_e = 0.001, \phi = 1.3$,
 $a = 0.5, M = 0.007, S_t = 1.7$.

x	v_{HPM}	Residual v_{HPM}
0	1.	$4.960761463384852 \times 10^{-10}$
0.1	0.8384982950365982	$4.267880216275103 \times 10^{-10}$
0.2	0.6939969486968401	$3.793644074653348 \times 10^{-10}$
0.3	0.5664959012626427	$3.489101326076564 \times 10^{-10}$
0.4	0.4559951000416662	$3.311324492168103 \times 10^{-10}$
0.5	0.3624944993672802	$3.223243623597471 \times 10^{-10}$
0.6	0.28599406059853705	$3.193621727777349 \times 10^{-10}$
0.7	0.22649375212014866	$3.19696804701524 \times 10^{-10}$
0.8	0.1839935493424646	$3.213469123634382 \times 10^{-10}$
0.9	0.15849343470146104	$3.229022028009403 \times 10^{-10}$
1	0.14999339765872777	$3.23514770528277 \times 10^{-10}$

Table 3. Absolute difference between OHAM and HPM solution keeping $W_e = 0.001$,
 $\phi = 1.3, a = 0.5, M = 0.007, S_t = 1.7$.

	v_{OHAM}	v_{HPM}	Absolute difference
0	1.	1.	0.
0.1	0.8384982947890038	0.8384982950365982	$2.475944727020618 \times 10^{-10}$
0.2	0.6939969482537215	0.6939969486968401	$4.431186528819353 \times 10^{-10}$
0.3	0.5664959006674037	0.5664959012626427	$5.952390247365713 \times 10^{-10}$
0.4	0.45599509933006155	0.4559951000416662	$7.116047751054566 \times 10^{-10}$
0.5	0.3624944985684351	0.3624944993672802	$7.988451011797793 \times 10^{-10}$
0.6	0.2859940597359656	0.28599406059853705	$8.625714587040534 \times 10^{-10}$
0.7	0.22649375121277338	0.22649375212014866	$9.073752860189188 \times 10^{-10}$
0.8	0.18399354840563786	0.1839935493424646	$9.368267273046627 \times 10^{-10}$
0.9	0.15849343374798386	0.15849343470146104	$9.534772971164784 \times 10^{-10}$
1	0.14999339669987022	0.14999339765872777	$9.588575489161144 \times 10^{-10}$

7 | SHEAR STRESS, VORTICITY, VOLUMETRIC FLOW RATE, AVERAGE VELOCITY

7.1. Shear Stress

The shear stress that is applied to the belt's outside is produced by

$$T_{xy}|_{x=0} = \mu \frac{dv}{dx}|_{x=0}. \quad (41)$$

By substituting the value, we have

$$T_{xy(OHAM)} = \mu \left(0. + \frac{1}{840} (-840S_t + 0.9999873149863626(70M^2(-12 + 4S_t) + 28(-1 + a^2)S_t^2(20M^2 + (-4M^2 - 10(-1 + \phi)(2 + 3\phi))S_t)W_e^2 - (-1 + a^2)^2S_t^4(168M^2 + (-24M^2 - 168(-1 + \phi^2))S_t)W_e^4)) \right). \quad (42)$$

$$T_{xy(HPM)} = \mu \left(-M^2 + \frac{1}{5040} S_t(-420(12 - 4M^2) + 168(-1 + a^2)S_t(20M^2 + (-4M^2 - 10(-1 + \phi)(2 + 3\phi))S_t)W_e^2 + S_t(-10080a^4\phi^2 + S_t(7560a^4\phi^2 + S_t(-1008(-1 + a^2)^2M^2 - 2520a^4\phi^2 + (a^4(-1008 + 144M^2 + 315\phi^2) - 6(-24M^2 - 168(-1 + \phi^2)) - 12a^2(24M^2 + 168(-1 + \phi^2)))S_t)))W_e^4) \right). \quad (43)$$

7.2. Vorticity

In lifting solving problems, Vorticity is

$$\vec{\omega} = \nabla \times \mathbf{V} = \frac{dv}{dx} \hat{\mathbf{k}}. \quad (44)$$

When we substitute the value, we get

$$\begin{aligned} \vec{\omega}_{(OHAM)} = \hat{\mathbf{k}} & (-0.0005952305446347397(-2 + x)x(70M^2(2 - 2x)S_t + 28(-1 + a^2)S_t^2(10M^2(-2 + 2x) + (M^2((-2 + x)x(2(-2 + x) + 2x) + (-2 + x)(1 + 2(-2 + x)x) + x(1 + 2(-2 + x)x)) - 5(-2 + 2x)(-1 + \phi)(2 + 3\phi))S_t)W_e^2 - (-1 + a^2)^2S_t^4(56M^2(-1 + 2x)(3 + (-3 + x)x) + 56M^2(-3 + 2x)(1 + (-1 + x)x) + (M^2((-2 + x)x((-2 + x)x(15(-2 + x) + 15x) + (-2 + x)(32 + 15(-2 + x)x) + x(32 + 15(-2 + x)x)) + (-2 + x)(6 + (-2 + x)x(32 + 15(-2 + x)x)) + x(6 + (-2 + x)x(32 + 15(-2 + x)x))) - 56(-1 + 2x)(3 + (-3 + x)x)(-1 + \phi^2) - 56(-3 + 2x)(1 + (-1 + x)x)(-1 + \phi^2))S_t)W_e^4) + \frac{1}{1680} (-2 + x)(840S_t - 0.9999873149863626(70M^2(-12 + (4 - (-2 + x)x)S_t) + 28(-1 + a^2)S_t^2(10M^2(2 + (-2 + x)x) + (M^2(-4 + (-2 + x)x(1 + 2(-2 + x)x)) - 5(2 + (-2 + x)x)(-1 + \phi)(2 + 3\phi))S_t)W_e^2 - (-1 + a^2)^2S_t^4(56M^2(3 + (-3 + x)x)(1 + (-1 + x)x) + (M^2(-24 + (-2 + x)x(6 + (-2 + x)x(32 + 15(-2 + x)x))) - 56(3 + (-3 + x)x)(1 + (-1 + x)x)(-1 + \phi^2))S_t)W_e^4)) + \frac{1}{1680} x(840S_t - 0.9999873149863626(70M^2(-12 + (4 - (-2 + x)x)S_t) + 28(-1 + a^2)S_t^2(10M^2(2 + (-2 + x)x) + (M^2(-4 + (-2 + x)x(1 + 2(-2 + x)x)) - 5(2 + (-2 + x)x)(-1 + \phi)(2 + 3\phi))S_t)W_e^2 - (-1 + a^2)^2S_t^4(56M^2(3 + (-3 + x)x)(1 + (-1 + x)x) + (M^2(-24 + (-2 + x)x(6 + (-2 + x)x(32 + 15(-2 + x)x))) - 56(3 + (-3 + x)x)(1 + (-1 + x)x)(-1 + \phi^2))S_t)W_e^4)))). \end{aligned} \quad (45)$$

$$\begin{aligned} \vec{\omega}_{(HPM)} = \hat{\mathbf{k}} & \left(-\frac{1}{1680} (-1 + x) \left(-1680M^2 - 280(6 + M^2(-2 + (-2 + x)x))S_t + 168(-1 + x)(-2(-1 + a^2)^2M^2(-1 + x)^3 + 5a^4(1 + x(2 + (-3 + x)(-1 + x)x))\phi^2)S_t^4W_e^4 - 3(-1 + x)(8(-1 + a^2)^2(-1 + x)^3(14 + M^2(-2 + 5(-2 + x)x)) - 7(16(-1 + x)^3 - 32a^2(-1 + x)^3 + 5a^4(-1 + (-2 + x)x)(1 + (-2 + x)^2x^2))\phi^2)S_t^5W_e^4 + 1120(-1 + x)S_t^2W_e^2((-1 + a^2)M^2(-1 + x) + 3a^4\phi^2W_e^2) + 56(-1 + x)S_t^3W_e^2(2(-1 + a^2)(-1 + x)(M^2(-2 + 3(-2 + x)x) + 5(2 + \phi - 3\phi^2)) + 45a^4(-1 + (-2 + x)x)\phi^2W_e^2) \right) \right). \end{aligned}$$

(46)

7.3. Volumetric Flow Rate

The following formula can be used to determine the volumetric flow rate of the withdrawal process

$$Q = \int_0^1 v dx. \quad (47)$$

When we substitute the value, we get

$$\begin{aligned} Q_{(\text{OHAM})} = & 1. - 0.3333291049954542M^2 + (-0.33333333333333337 + \\ & 0.13333164199818168M^2)S_t + (-0.1333316419981817 + 0.1333316419981817a^2)M^2S_t^2W_e^2 + \\ & (-0.1333316419981817 + 0.03809475485662334M^2 - 0.06666582099909085\phi + \\ & 0.19999746299727245\phi^2 + a^2(0.1333316419981817 - 0.03809475485662334M^2 + \\ & 0.06666582099909085\phi - 0.19999746299727245\phi^2))S_t^3W_e^2 + (-0.028571066142467523 + \\ & 0.057142132284935046a^2 - 0.028571066142467523a^4)M^2S_t^4W_e^4 + (-0.028571066142467523 + \\ & 0.006349125809437215M^2 + 0.028571066142467523\phi^2 + a^2(0.057142132284935046 - \\ & 0.01269825161887443M^2 - 0.057142132284935046\phi^2) + a^4(-0.028571066142467523 + \\ & 0.006349125809437215M^2 + 0.028571066142467523\phi^2))S_t^5W_e^4. \end{aligned}$$

(48)

$$\begin{aligned} Q_{(\text{HPM})} = & 1 - \frac{M^2}{3} + \frac{1}{5040}S_t(336(-5 + 2M^2) + S_tW_e^2(48(-1 + a^2)(14M^2 + (-4M^2 + 7(2 + \phi - \\ & 3\phi^2))S_t) + (-2520a^4\phi^2 + S_t(2520a^4\phi^2 + S_t(-144(-1 + a^2)^2M^2 - 945a^4\phi^2 + 2(8(-1 + a^2)^2(-9 + \\ & 2M^2) + 9(8 - 16a^2 + 7a^4)\phi^2)S_t)))W_e^2)). \end{aligned}$$

(49)

7.4. Average Velocity

The dimensional representation of the average velocity in a lifting problem-solving is expressed as

$$\bar{V} = \frac{Q}{\delta}. \quad (50)$$

In non-dimensional situations, the average velocity and flow rate are the same; therefore

$$\begin{aligned} \bar{V}_{(\text{OHAM})} = & 1. - 0.3333291049954542M^2 + (-0.33333333333333337 + \\ & 0.13333164199818168M^2)S_t + (-0.1333316419981817 + 0.1333316419981817a^2)M^2S_t^2W_e^2 + \\ & (-0.1333316419981817 + 0.03809475485662334M^2 - 0.06666582099909085\phi + \\ & 0.19999746299727245\phi^2 + a^2(0.1333316419981817 - 0.03809475485662334M^2 + \\ & 0.06666582099909085\phi - 0.19999746299727245\phi^2))S_t^3W_e^2 + (-0.028571066142467523 + \\ & 0.057142132284935046a^2 - 0.028571066142467523a^4)M^2S_t^4W_e^4 + (-0.028571066142467523 + \\ & 0.006349125809437215M^2 + 0.028571066142467523\phi^2 + a^2(0.057142132284935046 - \\ & 0.01269825161887443M^2 - 0.057142132284935046\phi^2) + a^4(-0.028571066142467523 + \\ & 0.006349125809437215M^2 + 0.028571066142467523\phi^2))S_t^5W_e^4. \end{aligned}$$

(51)

and

$$\bar{V}_{(HPM)} = 1 - \frac{M^2}{3} + \frac{1}{5040} S_t (336(-5 + 2M^2) + S_t W_e^2 (48(-1 + a^2)(14M^2 + (-4M^2 + 7(2 + \phi - 3\phi^2))S_t) + (-2520a^4\phi^2 + S_t(2520a^4\phi^2 + S_t(-144(-1 + a^2)^2M^2 - 945a^4\phi^2 + 2(8(-1 + a^2)^2(-9 + 2M^2) + 9(8 - 16a^2 + 7a^4)\phi^2)S_t)))W_e^2)) . \quad (52)$$

The equations of velocity and shear stress, etc., are essential for examining fluid motion and its interaction with surfaces. Because of viscosity and boundary conditions, fluid particles' speeds inside a flow field frequently vary with position, as described by velocity equations. The internal frictional forces in a fluid caused by velocity gradients, especially in viscous flows, are measured using shear stress equations. Understanding fluid behavior is crucial for maximizing performance and guaranteeing structural integrity in engineering applications like pipe flow, lubrication systems, and aerodynamics.

7 | RESULTS AND DISCUSSION

This paper investigates Johnson-Segalman fluid's magneto-hydrodynamic (MHD) thin-film flow in connection to lifting problems on vertical surfaces. For each problem, the relevant boundary conditions and differential equations are developed. The approximate analytical solutions for problems are derived using OHAM and HPM.

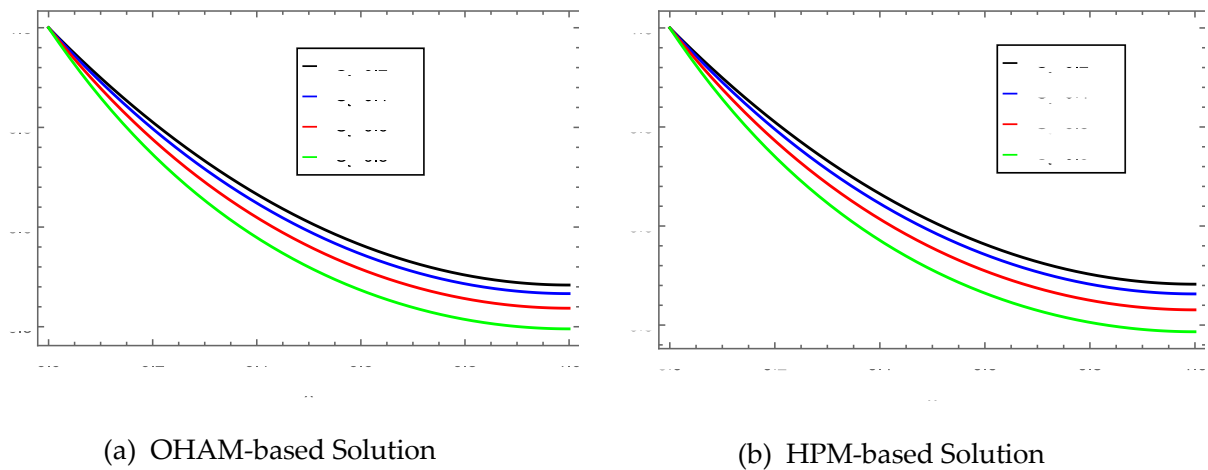
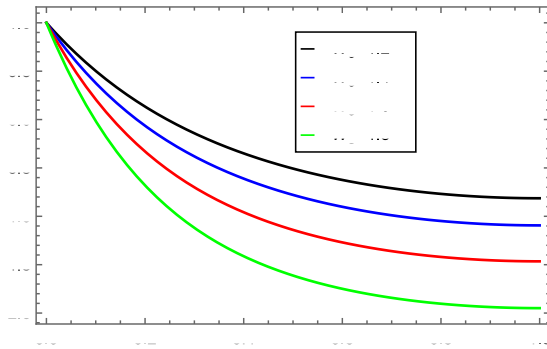
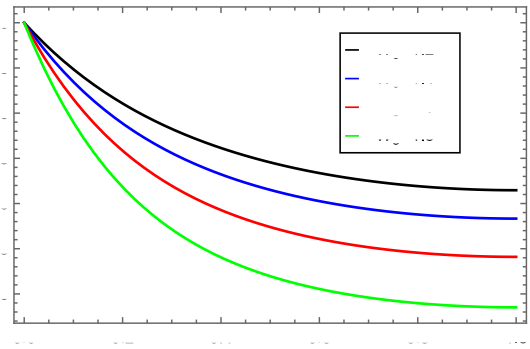


Figure 2. The effect of S_t on the velocity $v(x)$ on the withdrawal case for

$$W_e = 1.2; \phi = 1; a = 0.5; M = 1.6.$$



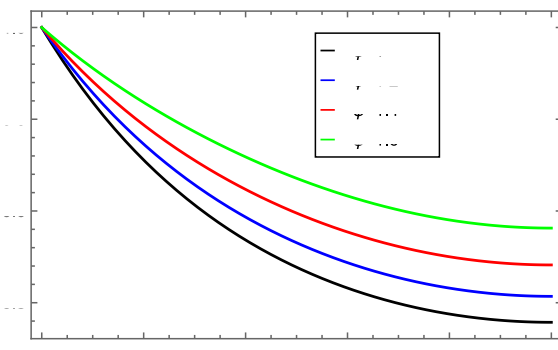
(a) OHAM-based Solution



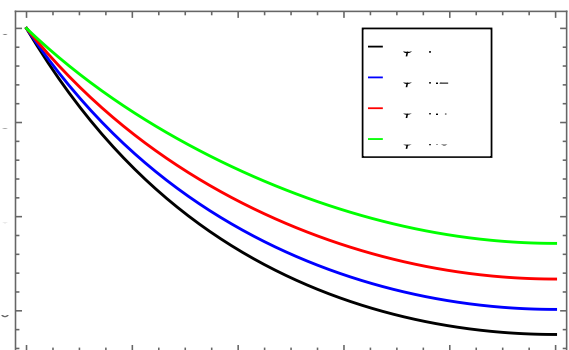
(b) HPM-based Solution

Figure 3. The effect of W_e on the velocity $v(x)$ of the withdrawal case for

$$W_e = 1.8; S_t = 1.2; \phi = 1; a = 0.5; M = 1.6.$$



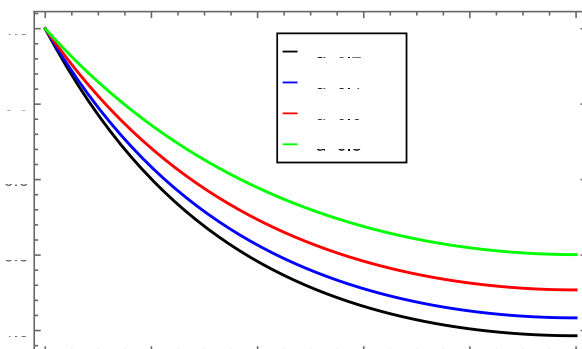
(a) OHAM-based Solution



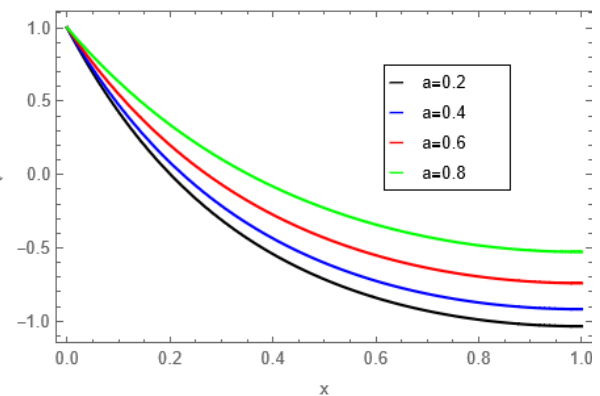
(b) HPM-based Solution

Figure 4. The effect of ϕ on the velocity $v(x)$ of the withdrawal case for

$$S_t = 1.2; W_e = 1; a = 0.5; M = 1.6;.$$



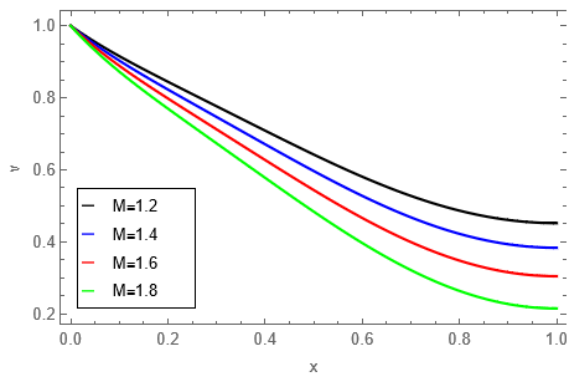
(a) OHAM-based Solution



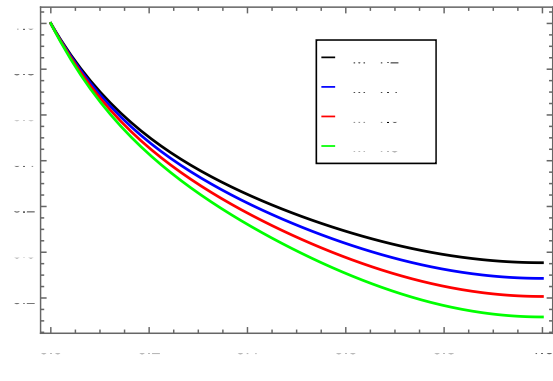
(b) HPM-based Solution

Figure 5. The effect of a on the velocity $v(x)$ of the withdrawal case for

$$S_t = 1.2; W_e = 1; \phi = 0.5; M = 1.6;.$$

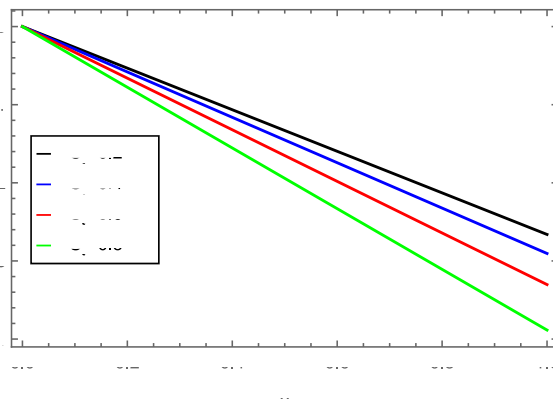


(a) OHAM-based Solution

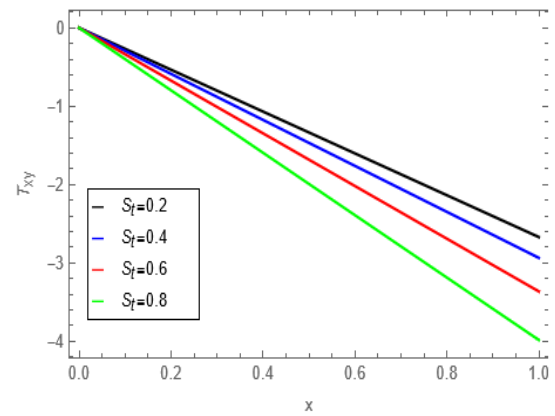


(b) HPM-based Solution

Figure 6. The effect of M on velocity $v(x)$ of the withdrawal case for $S_t = 1.2; W_e = 1; \phi = 0.5; a = 1.6;$

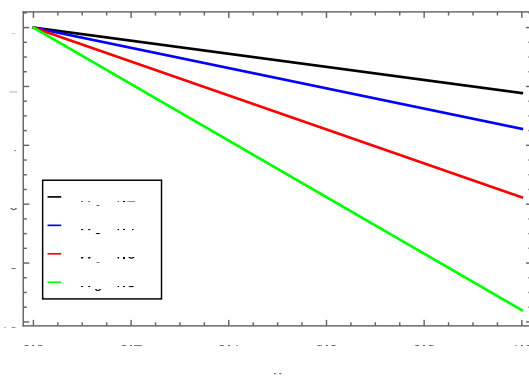


(a) OHAM-based Solution

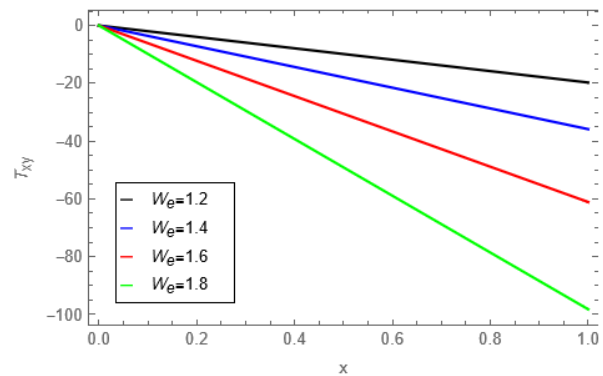


(b) HPM-based Solution

Figure 7. The effect of S_t on shear stress of the withdrawal case for $W_e = 1.2; \phi = 1; a = 0.5; M = 1.6;$

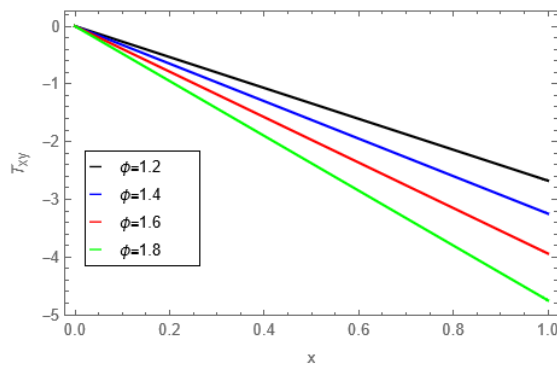


(a) OHAM-based Solution



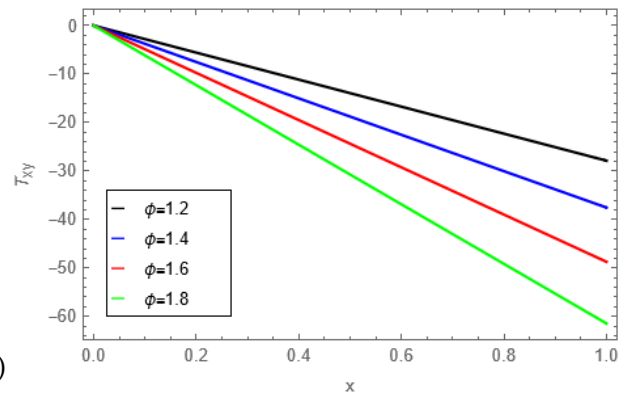
(b) HPM-based Solution

Figure 8. The effect of W_e on shear stress of the withdrawal case for $S_t = 1.2; \phi = 1; a = 1.5; M = 1.6;$



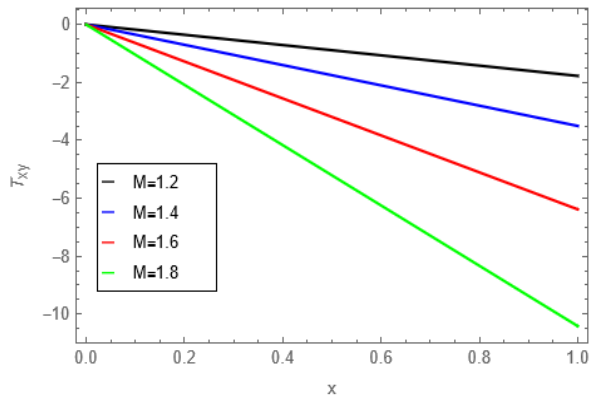
OHAM-based Solution

(a)



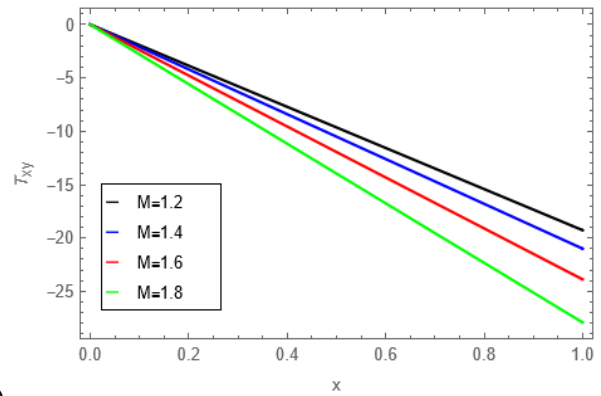
(b) HPM-based Solution

Figure 9. The effect of ϕ on the shear stress of the withdrawal case for
 $S_t = 1.2; W_e = 1.2, a = 1.5; M = 1.6;$



OHAM-based Solution

(a)



(b) HPM-based Solution

Figure 10. The effect of M on the shear stress of the withdrawal case for
 $S_t = 1.2; W_e = 1.2; \phi = 1; a = 1.5;$

In this current study, we calculated approximate solutions to a nonlinear differential equation containing various parameters, including S_t , W_e , ϕ and a . Tables 1 and 2 highlight the velocity profile obtained by OHAM and HPM, along with their residuals. Table 3 shows the absolute difference between solutions obtained by OHAM and HPM. Figures 2 and 3 show that increasing the slip parameter. S_t and Weissenberg number W_e decreases the velocity profile due to enhanced surface resistance and fluid elasticity, respectively. Figures 4 and 5 indicate that an increase in the parameter ϕ and slip parameter a promotes higher velocities, as these parameters facilitate smoother fluid movement along the film. Figure 6 reveals that the magnetic parameter M has a damping effect on velocity due to the opposing Lorentz force. Figure 7 displays the result of the parameter S_t on the withdrawal problem's shear stress. Figure 8 displays the result of the parameter W_e on the withdrawal problem's shear stress. Figure 9 displays the effect of parameter ϕ on the withdrawal problem's shear

stress. Figure 10 displays the effect of parameter M on the withdrawal problem's shear stress, and an inverse relation is shown between S_t , W_e , ϕ , M , and shear stress profile.

8 | CONCLUSION

In this study, a nonlinear differential equation was effectively solved using the Homotopy Perturbation Method (HPM) and the Optimal Homotopy Asymptotic Method (OHAM). The efficiency and precision of both approaches in capturing the nonlinear behavior of the system were demonstrated by the derivation of analytical formulas for velocity, flow rate, shear stress, and vorticity.

The findings provide insightful information about the flow properties and a helpful analytical basis for additional theoretical research or real-world engineering applications. Future research might concentrate on applying these techniques to more intricate geometries or adding more physical phenomena like magnetic fields, thermal transfer, or unsteadiness. Thus, a potential paradigm for addressing a broad class of nonlinear issues in fluid dynamics and beyond is presented by the combination of OHAM and HPM.

These findings are widely applied in engineering and physical sciences to model complex systems with high accuracy while avoiding the difficulties of purely numerical methods. The results derived from these methods, such as velocity profiles, shear stress, and vorticity, have significant real-world applications, particularly in fluid dynamics, material science, and thermal engineering. Future work can explore more complex boundary conditions, additional physical effects, or numerical validation of the results.

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REFERENCES

- [1] L. D. Landau, "Fluid mechanics, course of theoretical physics," *Addison-Wesley Publ. Co*, vol. 6, 1959.
- [2] M. K. Alam, M. T. Rahim, E. J. Avital, S. Islam, A. M. Siddiqui, and J. J. R. Williams, "Solution of the steady thin film flow of non-Newtonian fluid on vertical cylinder using Adomian Decomposition Method," *J. Frankl. Inst.*, vol. 350, no. 4, pp. 818–839, May 2013, doi: 10.1016/j.jfranklin.2013.01.010.
- [3] I. J. Rao, "Flow of a Johnson–Segalman fluid between rotating co-axial cylinders with and without suction," *Int. J. Non-Linear Mech.*, vol. 34, no. 1, pp. 63–70, Jan. 1999, doi: 10.1016/S0020-7462(97)00081-4.
- [4] I. J. Rao and K. R. Rajagopal, "Some simple flows of a Johnson–Segalman fluid," *Acta Mech.*, vol. 132, no. 1–4, pp. 209–219, Mar. 1999, doi: 10.1007/BF01186968.
- [5] H. Ashraf, S. Sabir, A. M. Siddiqui, H. U. Rehman, B. Almutairi, and N. A. Shah, "Heat transfer analysis of temperature dependent viscosity Johnson–Segalman fluid film flow on a vertical heated belt," *Case Stud. Therm. Eng.*, vol. 49, p. 103362, Sep. 2023, doi: 10.1016/j.csite.2023.103362.
- [6] M. Qayyum, M. Faisal, S. Rezapour, and M. Inc, "Numerical solutions of thin film flows of non-Newtonian fluids via optimal homotopy asymptotic approach," *Mod. Phys. Lett. B*, vol. 38, no. 35, p. 2450352, Dec. 2024, doi: 10.1142/S0217984924503524.
- [7] H. Marasi, M. Daneshbastam, and A. Alizadeh, "Analytic investigation of steady thin film flow of non-newtonian fluid on vertical cylinder for lifting and drainage problems," vol. 11, no. 4, 2021.
- [8] N. A. Khan, F. S. Alshammari, C. A. Tavera Romero, M. Sulaiman, and S. Mirjalili, "An Optimistic Solver for the Mathematical Model of the Flow of Johnson Segalman Fluid on the Surface of an Infinitely Long Vertical Cylinder," *Materials*, vol. 14, no. 24, p. 7798, Dec. 2021, doi: 10.3390/ma14247798.
- [9] M. Hameed and R. Ellahi, "Thin film flow of non-Newtonian MHD fluid on a vertically moving belt," *Int. J. Numer. Methods Fluids*, vol. 66, no. 11, pp. 1409–1419, Aug. 2011, doi: 10.1002/fld.2320.
- [10] M. K. Alam, A. M. Siddiqui, M. T. Rahim, and S. Islam, "Thin-film flow of magnetohydrodynamic (MHD) Johnson–Segalman fluid on vertical surfaces using the Adomian decomposition method," *Appl. Math. Comput.*, vol. 219, no. 8, pp. 3956–3974, Dec. 2012, doi: 10.1016/j.amc.2012.10.032.
- [11] R. W. Kolkka, D. S. Malkus, M. G. Hansen, and G. R. Ierley, "Spurt phenomena of the Johnson–Segalman fluid and related models," *J. Non-Newton. Fluid Mech.*, vol. 29, pp. 303–335, 1988, doi: 10.1016/0377-0257(88)85059-6.
- [12] H. Alahmadi and R. Nawaz, "Analyzing the influences of induced magnetic fields on heat and mass transfer during peristaltic motion of Johnson–Segalman fluid," *J. Therm. Anal. Calorim.*, vol. 149, no. 14, pp. 7593–7603, Jul. 2024, doi: 10.1007/s10973-024-13372-5.
- [13] H. Zahir, J. Akram, Mehnaz, R. K. Alhefthi, R. Fatima, and M. Inc, "Effect of nonlinear thermal radiation and homogeneous-heterogeneous reactions on peristaltic propulsion of Johnson–Segalman fluid," *Ain Shams Eng. J.*, vol. 15, no. 2, p. 102383, Feb. 2024, doi: 10.1016/j.asej.2023.102383.
- [14] K. B. Migler, G. Massey, I. Hervet, and L. Leger, "The slip transition at the polymer-solid interface," *J. Phys. Condens. Matter*, vol. 6, no. 23A, pp. A301–A304, Jun. 1994, doi: 10.1088/0953-8984/6/23A/049.

- [15] T. Hayat, Y. Wang, A. M. Siddiqui, and K. Hutter, "Peristaltic motion of a Johnson-Segalman fluid in a planar channel," *Math. Probl. Eng.*, vol. 2003, no. 1, pp. 1–23, Jan. 2003, doi: 10.1155/S1024123X03308014.
- [16] G. Adomian, "A review of the decomposition method and some recent results for nonlinear equations," *Math. Comput. Model.*, vol. 13, no. 7, pp. 17–43, 1990, doi: 10.1016/0895-7177(90)90125-7.
- [17] G. Adomian, *Solving frontier problems of physics: the decomposition method*, vol. 60. Springer Science & Business Media, 2013.
- [18] A. M. Siddiqui, M. Hameed, B. M. Siddiqui, and Q. K. Ghori, "Use of Adomian decomposition method in the study of parallel plate flow of a third grade fluid," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 15, no. 9, pp. 2388–2399, Sep. 2010, doi: 10.1016/j.cnsns.2009.05.073.
- [19] E. Alizadeh, K. Sedighi, M. Farhadi, and H. R. Ebrahimi-Kebria, "Analytical approximate solution of the cooling problem by Adomian decomposition method," *Commun. Nonlinear Sci. Numer. Simul.*, vol. 14, no. 2, pp. 462–472, Feb. 2009, doi: 10.1016/j.cnsns.2007.09.008.
- [20] M. Farooq, A. Khan, R. Nawaz, S. Islam, M. Ayaz, and Y.-M. Chu, "Comparative study of generalized couette flow of couple stress fluid using optimal homotopy asymptotic method and new iterative method," *Sci. Rep.*, vol. 11, no. 1, p. 3478, Feb. 2021, doi: 10.1038/s41598-021-82746-8.
- [21] A. M. Siddiqui, R. Mahmood, and Q. K. Ghori, "Thin Film Flow of a Third Grade Fluid on a Moving Belt by He's Homotopy Perturbation Method," *Int. J. Nonlinear Sci. Numer. Simul.*, vol. 7, no. 1, Jan. 2006, doi: 10.1515/IJNSNS.2006.7.1.7.