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A Decision Support System for Assessment of Digital Marketing Platform Selection Using Novel Circular Intuitionistic Fuzzy Dombi Aggregation Operators

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ABSTRACT

Social media services such as Facebook, YouTube, Instagram, and Telegram are an essential part of digital marketing, as they increase target reach, engagement, and brand exposure. Nevertheless, it is hard to measure their effectiveness because of different user behaviors, the format of the content, and ever-changing algorithms. The idea of circular intuitionistic fuzzy set (Cir-IFS) and Dombi operations are strong tools to handle doubt and ambiguity in multi-attribute decision-making (MADM) issues. This study established two novel aggregation operators (AOs), namely circular intuitionistic fuzzy Dombi prioritized weighted averaging (CIFDPWA) and circular intuitionistic fuzzy Dombi prioritized weighted geometric (CIFDPWG) operators. The mathematical properties of established AOs, such as idempotency, boundedness, and monotonicity, are properly established. In order to justify the proposed model, a real-life case study about how to choose the appropriate social media platform when it comes to carrying out digital marketing will be discussed. The proposed theory is a suitable tool for a precise investigation of MADM problems, such as assessing platforms like Facebook, YouTube, Instagram, and Telegram against numerous attributes, including user engagement, algorithmic fairness, data privacy, and usability. The established AOs are applied to rank the alternatives, revealing their efficiency in solving the complexities of the modern digital decision-making framework. Furthermore, the framework of Cir-IFS surpasses our diagnosed theory by incorporating circular degree, whereas simple IFS fails to aggregate information that includes circular degree. To confirm the validity of the established model, we compare it with existing models. Then, we deliberate on a solid conclusion.

Keywords: Circular intuitionistic fuzzy set, Dombi operations, multi-attribute decision-making, social media platforms, aggregation operators

(MSC2020) Codes: 03E72, 03B52, 90B50, 91D30, 03E72

1 | INTRODUCTION

Previously, companies struggled to utilize social media effectively due to a lack of knowledge about digital tools, insufficient data analysis, and an inability to accurately gauge the success of their campaigns. Due to a lack of knowledge, many small businesses placed great reliance on traditional advertisements without realizing the scale and reach that social media had to offer in terms of available spots and depth of interest. In today's reality, as social network consumption has been increasing regularly, new challenges have emerged in the form of data privacy issues, the need to be aware of algorithm alterations, and the importance of selecting the most suitable platform in the vast virtual world. The wealth of options available late and the intricacy of user trends imply that more sophisticated, data-driven choices are needed to maximize the effectiveness of electronic marketing efforts. In order to demonstrate the relevance of the suggested model of aggregation, we provide a real-life case study carried out on the choice of the most appropriate social media to be used in digital marketing in business. We evaluate four alternatives, like Facebook, YouTube, Instagram, and Telegram, based on attributes such as user engagement, algorithmic fairness, privacy, data security, and platform usability. The case replicates a real-life decision-making challenge for businesses seeking cost-effective, data-driven, and customer-centric solutions in an increasingly digital marketplace. Our established technique helps decision-makers rank alternatives by quantifying doubt and collecting preferences.

1.1 | Fuzzy Set and Extension

[1] Introduced the model of fuzzy set (FSs) to address conditions that include ambiguity and hesitation in data, which traditional binary logic cannot effectively handle. The classical set model functions on a binary notion, where elements either belong to a collection or do not, making it challenging to apply the theory to real-world conditions that frequently involve vagueness. The FS permits membership $ȷ(\chi)$, where a degree can belong to a collection to a definite degree, ranging from $[0, 1]$. Such flexibility allows FSs to be highly appropriate in applications like pattern recognitions, decision-making, control systems, and artificial intelligence, where certain conclusions need to be generated on incorrect or poor information. Later on, [2] offered an advanced model of an intuitionistic fuzzy set (IFS) by using the idea of FSs in two different degrees, such as membership grade (MG) $ȷ(\chi)$ and membership grade (NMG) $œ(\chi)$, which are limited by a closed interval [0, 1] and satisfy the condition $0 \leq ȷ(\chi) + œ(\chi) \leq 1$. The theory of IFS is extended into the Cir-IFS by [3]. These extensions provide greater roughness by incorporating degrees of MG, NMG, and radius grade (RG), which are crucial when handling ambiguous or incompatible data. Cir-IFSs present a precise, circular illustration to determine uncertainty more methodically. This circularity suggests a strong

mathematical framework that supports more flexible modelling of human reasoning patterns under incomplete information.

1.2 | Multi-Attribute Decision-Making

The MADM is a process that is utilized to assess and select between choices that are described using several often-contradictory criteria or attributes. Therefore, many studies propose using various MADMs, with each operating on a different structure, such as the dynamic intuitionistic fuzzy MADM presented by [4] and MADM with q-rung picture fuzzy information introduced by [5] and MADM using an Archimedean aggregation operator in a T-spherical fuzzy environment, investigated by [6]. Ganie and Singh presented a new picture fuzzy distance measure and a variant of MADM methodology [7]. Hamacher aggregation operators for Pythagorean fuzzy sets and their application in MADM were explained by [8]. The idea of MADM using the q-rung orthopair fuzzy set-based information discussed by [9] and the concept of confidence level-based AOs under the T-spherical fuzzy offered by [10]. T-spherical fuzzy power aggregation operators and their applications in MADM were defined by [11]. The methods of MADM, pattern recognition, and clustering based on T-spherical fuzzy information measures were put forward by [12]. The concept of MADM under the T-spherical fuzzy sets discussed by [13] and the solution of the construction material selection problem using the MADM approach provided by [14]. [15] offered the theory of prioritized AOs under the MADM method and the idea of Aczel-Alsina operations for the solution of MADM problems given by [16].

1.3 | Dombi Aggregation Operator

The Dombi AOs are nonlinear, parameterized TNoM and TCNoM that suggest notable flexibility in aggregating data. Unlike inflexible operators like min and product, Dombi operators can moderate the level of conjunction or disjunction through adaptable parameters. This variation allows for a more advanced combination of involvement information, which is particularly suitable when models measure changes in preference or similarity.

Dombi TNoM and TCNoM are flexible operations that have application in a fuzzy set theory to address complex situations of decision making. They have an adaptable parameter that enables their adaptive control of intersection and union operations, which could also be extremely useful for dealing with uncertainty and vagueness. This flexibility increases their suitability for the MADM and the fuzzy set theory.

There is indeed an advantage of adopting Dombi operators in a fuzzy scenario that adds adaptability and realism to the purpose of aggregating, making them applicable in numerous MADM applications. Many researchers use Dombi AOs based on a fuzzy framework for many fields of application. [17] offered an application in healthcare waste management based on Dombi AOs. [18]

established a model for the assessment of typhoon disasters based on Dombi AOs. [19] proposed Dombi AOs under IFS, [20] proposed Dombi AOs based on intuitionistic fuzzy rough power AOs, and [21] established Dombi AOs for the application of an agricultural robotics system.

1.4 | Motivation and Principal Findings

In recent years, numerous mathematicians have endeavored to develop multiple approaches for assessing decision-making problems within various fuzzy environments. For example, the theory of IFS is a reliable and trustworthy tool for aggregation of complex real-life problems in terms as the values of MG and NMG are observed to be in the range of a closed interval [0,1]. However, the theory of IFS is not suitable for assessing Cir-IFS-based data. We have investigated the Dombi operations for Cir-IFS-based information. By drawing motivation from the concept of Cir-IFS and Dombi operational laws, we have constructed the new theories, which are discussed as follows:

1. We have developed the idea of Cir-IFDPWA and Cir-IFDPWG operators.
2. We have discussed some fundamental axioms of AOs.
3. We presented the MADM algorithm on the basis of the postulated theory.
4. Presented the case study on the selection of a suitable social media platform.
5. Compare our proposed theory with other preset MADM approaches.

1.5 | Organization

This paper is structured as follows: Section 2 presents the foundational concepts of Cir-IFS, including definitions, score functions, Dombi AOs, and some operational laws. Section 3 presents the new AOs Cir-IFDPWA and Cir-IFDPWG operators and formally establishes their mathematical properties, such as idempotency, boundedness, and monotonicity. Section 4 presents the MADM algorithm and outlines the step-by-step process for applying the operators in real-world situations. Section 5 presents a case study on social media platform selection, presenting numerical outcomes and an in-depth investigation. Section 6 suggests a comparative assessment of the proposed operators, and Section 7 outlines the study with the conclusion and future directions. The graphically represented structure of this paper is displayed in Figure 1.

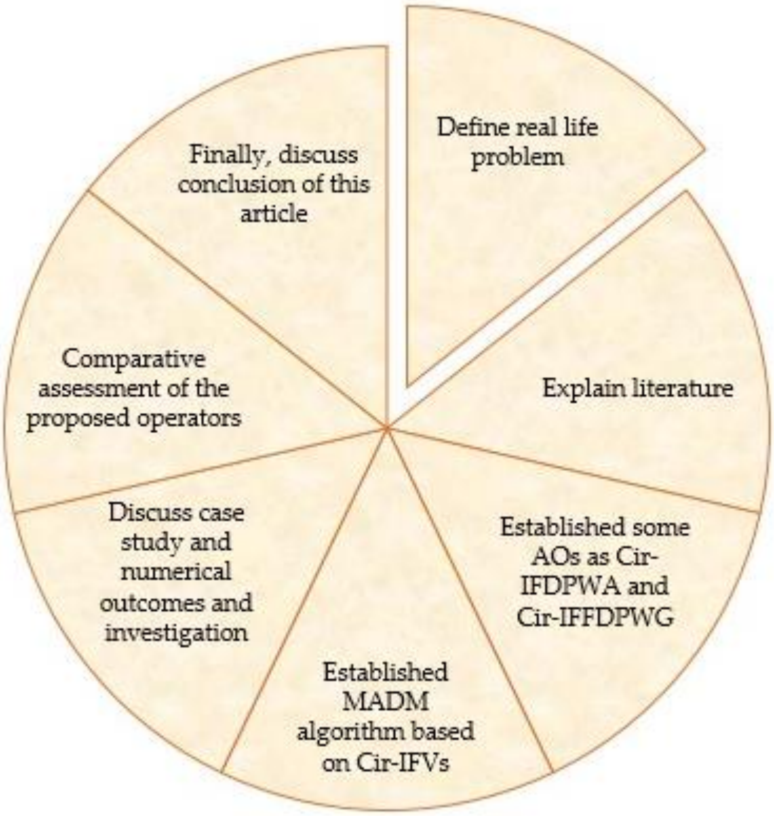

Figure 1. Schematic of the structure of this article

2 | PRELIMINATRIES

In this section, we will discuss some fundamental concepts related to diagnosed AOs that will provide a better understanding.

Definition 1. [22] Let X^o be a non-empty set. A circular intuitionistic fuzzy value (Cir-IFV) set B is explained as follows:

$$\mathrm{B} = \{ \langle\, \chi, \jmath(\chi), \text{œ}(\chi), \check{y}(\chi) \rangle : \chi \in X^o \}$$

where $\jmath(\chi): X^o \rightarrow [0, 1]$ and $\text{œ}(\chi): X^o \rightarrow [\,0, 1]$ act for MG and NMG of the member $\chi \in X$ to the set B, respectively, $\check{y}(\chi) \in \left[0, \sqrt{2}\right]$ is the circular degree of each element of the set around the circle, and also fulfilling the condition $0 \leq \jmath(\chi) + \text{œ}(\chi) \ \leq 1$. The hesitancy value is indicated by $\pi(\chi) = 1 - \jmath(\chi) + \text{œ}(\chi)$. For easiness, $\left(\jmath(\chi), \text{œ}(\chi), \check{y}(\chi)\right)$ called Cir-IF values (Cir-IFVs).

Definition 2. [23] Let $M = \{\langle\, \chi, \jmath(\chi), \text{œ}(\chi), \check{y}(\chi) \rangle : \chi \in X\,\}$ and $\mathbb{N}'' = \{\, \langle\, \chi, \jmath(\chi), \text{œ}(\chi), \check{y}(\chi) : \chi \in X\, \rangle\}$ be two Cr-IFVs in the sample space X and $\lambda > 0$ be a random real value, and the similar operations are explained as listed:

1. $\mathbb{M} \oplus \mathbb{N}'' = \left\{\chi, \left(\jmath_{\dot{M}}(\chi) + \jmath_{\dot{N}}(\chi) - \jmath_{\dot{M}}(\chi)\jmath_{\dot{N}}(\chi), \text{œ}_{\dot{M}}(\chi)\text{œ}_{\dot{N}}(\chi), \check{y}_{\dot{M}}(\chi) + \check{y}_{\dot{N}}(\chi) - \check{y}_{\dot{M}}(\chi)\check{y}_{\dot{N}}(\chi)\right) : \chi \in X\right\}$
2. $\mathbb{M} \otimes \mathbb{N}'' = \left\{\, \left(\, \chi, \jmath_{\dot{M}}(\chi)\jmath_{\dot{N}}(\chi), \text{œ}_{\dot{M}}(\chi) + \text{œ}_{\dot{N}}(\chi) - \text{œ}_{\dot{M}}(\chi)\text{œ}_{\dot{N}}(\chi), \check{y}_{\dot{M}}(\chi)\check{y}_{\dot{N}}(\chi)\right) \chi \in X\right\}$
3. $\lambda\mathbb{M} = \left\{\left(\, \chi,\ 1 - \left(\,1 - \jmath_{\dot{M}}(\chi)\right)^{\lambda}, \left(\text{œ}_{\dot{M}}(\chi)\right)^{\lambda}, 1 - \left(\,1 - \check{y}_{\dot{M}}(\chi)\right)^{\lambda}\right) : \chi \in X\right\}$
4. $\mathbb{M}^{\,\lambda} = \left\{\left(\, \chi, \left(\,\jmath_{\dot{M}}(\chi)\right)^{\lambda},\ 1 - \left(\,1 - \text{œ}_{\dot{M}}(\chi)\right)^{\lambda}, \left(\check{y}_{\dot{M}}(\chi)\right)^{\lambda} : \chi \in X\right\}$

The degree of the score function is to transform the Cir-IFS to a single scalar value, making comparison and ranking easier. It simplifies decision-making by allowing a comparative judgment of alternatives based solely on their desirability.

Definition 3. [24] Let $\tau = (ʝ_\tau, œ_\tau, \breve{y}_\tau)$ be any Cir-IFS. Then, the score function for the Cir-IFS $\dot{\alpha}$ is explained by:

$$\phi(\tau) = (ʝ_\tau - œ_\tau) \times \breve{y}_\tau \tag{1}$$

where $\phi(\tau) \in [-1,\ 1]$.

Definition 4. [25] Let $\tau = (ʝ_\tau, œ_\tau, \breve{y}_\tau)$ be any Cir-INF. Then, the accuracy function for the Cir-IFVs τ is explained by:

$$\psi(\tau) = (ʝ_\tau + œ_\tau) \times \breve{y}_\tau \tag{2}$$

where $\psi(\dot{\alpha}) \in [\,0,\ 1]$.

According to definitions 3 and 4, if $\tau = (ʝ_\tau, œ_\tau, \breve{y}_\tau)$ and $L' = (ʝ_{T'}, œ_{T'}, \breve{y}_{T'})$ be two Cir-IFVs, then:

1. If $\phi(\tau) > \phi(L')$, then $\tau > L'$.
2. If $\phi(\tau) < \phi(L')$, then $\tau < L'$.
3. If $\phi(\tau) = \phi(L')$, then:

- If $\boldsymbol{\psi}(\tau) > \boldsymbol{\psi}(L')$, then $\tau > L'$;
- If $\boldsymbol{\psi}(\tau) < \boldsymbol{\psi}(L')$, then $\tau < L'$;
- If $\boldsymbol{\psi}(\tau) = \boldsymbol{\psi}(L')$, then $\tau = L'$.

Definition 5. [26] Consider a^o and b^o be two real value. Then, Dombi TNoM and TCNoM are explained as:

$$Domb(a^o, b^o) = \frac{1}{1 + \left\{\left(\frac{1-a^o}{a^o}\right)^{\mathfrak{C}} + \left(\frac{1-b^o}{b^o}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \tag{3}$$

$$Domb^{\circ}(a^o, b) = 1 - \frac{1}{1 + \left\{\left(\frac{a^o}{1-a^o}\right)^{\mathfrak{C}} + \left(\frac{b^o}{1-b^o}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \tag{4}$$

where $\mathfrak{C} \geq 1$ and $(a^o, b^o) \in [0,1] \times [0,1]$.

3 | CIRCULAR INTUITIONISTIC FUZZY DOMBI AGGREGATION OPERATOR

In this section, we will first define operational laws for Cir-IFVs on behalf of Dombi TNoM and TCNoM, and then diagnose new AOs, specifically Cir-IFDPWA and Cir-IFDPWG operators. Additionally, it provides some axioms of AOs, including idempotency, boundedness, and monotonicity, which are relevant to the proposed work.

Definition 6. Let $\tau = (ꭍ_\tau, œ_\tau, ÿ_\tau)$, $T' = (ꭍ_{T'}, œ_{T'}, ÿ_{T'})$ be two Cir-IFVs and $\mathfrak{C} \geq 1$ and $\lambda > 0$ be absolute values, so Dombi TNoM and TCNoM operators of Cir-IFV are explained as:

i. $$\tau \oplus T' = \begin{pmatrix} 1 - \frac{1}{1+\left\{\left(\frac{ꭍ_\tau}{1-ꭍ_\tau}\right)^{\mathfrak{C}} + \left(\frac{ꭍ_{T'}}{1-ꭍ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, \\ \frac{1}{1+\left\{\left(\frac{1-œ_\tau}{œ_\tau}\right)^{\mathfrak{C}} + \left(\frac{1-œ_{T'}}{œ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, 1 - \frac{1}{1+\left\{\left(\frac{ÿ_\tau}{1-ÿ_\tau}\right)^{\mathfrak{C}} + \left(\frac{ÿ_{T'}}{1-ÿ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \end{pmatrix}$$

ii. $$\tau \otimes T' = \begin{pmatrix} \frac{1}{1+\left\{\left(\frac{1-ꭍ_\tau}{ꭍ_\tau}\right)^{\mathfrak{C}} + \left(\frac{1-ꭍ_{T'}}{ꭍ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, \\ 1 - \frac{1}{1+\left\{\left(\frac{œ_\tau}{1-œ_\tau}\right)^{\mathfrak{C}} + \left(\frac{œ_{T'}}{1-œ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, \frac{1}{1+\left\{\left(\frac{1-ÿ_\tau}{ÿ_\tau}\right)^{\mathfrak{C}} + \left(\frac{1-ÿ_{T'}}{ÿ_{T'}}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \end{pmatrix}$$

iii. $$\lambda\tau = \left(1 - \frac{1}{1+\left\{\lambda\left(\frac{ꭍ_\tau}{1-ꭍ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-œ_\tau}{œ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{ÿ_\tau}{1-ÿ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \right)$$

iv. $$\tau^\lambda = \left(\frac{1}{1+\left\{\lambda\left(\frac{1-ꭍ_\tau}{ꭍ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, 1 - \frac{1}{1+\left\{\lambda\left(\frac{1-œ_\tau}{œ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}}, \frac{1}{1+\left\{\lambda\left(\frac{1-ÿ_\tau}{ÿ_\tau}\right)^{\mathfrak{C}}\right\}^{\frac{1}{\mathfrak{C}}}} \right)$$

Definition 7. Let $\tau_i = (ꭍ_i, œ_i, ÿ_i)(i = 1, 2, \ldots\ldots, n)$, be a value of Cir-IFVs. So, the Cir-IFDPWA operator is a function. $\tau^n \to \tau$, we get:

$$Cir - IFDPWA(\tau_1, \tau_2, \ldots\ldots, \tau_n) = \frac{\Delta_1}{\sum_{i=1}^n \Delta_i}\tau_1 \oplus \frac{\Delta_2}{\sum_{i=1}^n \Delta_i}\tau_2 \oplus \ldots \oplus \frac{\Delta_2}{\sum_{i=1}^n \Delta_i}\tau_2$$

where $Ж_i = \frac{\Delta_i}{\sum_{i=1}^n \Delta_i}$, $\Delta_i = \prod_{i=1}^n \phi(\tau)$ and $\Delta_1 = 1$ form the weight vector (WV) of $\tau_i (i = 1,2, \ldots\ldots, n)$, $Ж_i > 0$, and $\sum_{i=1}^n Ж_i = 1$.

Theorem 1. Let $\tau_i = (ꭍ_i, œ_i, ÿ_i)(i = 1, 2, \ldots\ldots, n)$ be a collection of Cir-IFVs, and so, the combined outcomes by applying Cir-IFDWA are also Cir-IFVs:

$$Cir-IFDPWA\ (\tau_1,\tau_2,\ldots\ldots,\tau_n)=\oplus_{i=1}^{n}\text{Ж}_i\tau_i$$

$$=\left(\begin{array}{c} 1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\dfrac{\text{ʝ}_i}{1-\text{ʝ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}, \\ \dfrac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\dfrac{1-\text{œ}_i}{\text{œ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},1-\dfrac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\dfrac{\text{ў}_i}{1-\text{ў}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}} \end{array}\right) \qquad (5)$$

where $\text{Ж}_i=\frac{\Delta_i}{\sum_{i=1}^{n}\Delta_i}$, $\Delta_i=\prod_{i=1}^{n}\phi(\tau)$ and $\Delta_1=1$ form the weight vector (WV) of $\tau_i(i=1,2,\ldots.,n)$, $\text{Ж}_i>0$, and $\Sigma_{i=1}^{n}\text{Ж}_i=1$.

Proof. Now, we demonstrate Theorem 1 through the technique of induction.

Consider two Cir-IFVs

$$\tau_1=\left(\begin{array}{c} 1-\dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{\text{ʝ}_1}{1-\text{ʝ}_1}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}, \\ \dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{1-\text{œ}_1}{\text{œ}_1}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},1-\dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{\text{ў}_1}{1-\text{ў}_1}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}} \end{array}\right)$$

$$\tau_2=\left(\begin{array}{c} 1-\dfrac{1}{1+\left\{\text{Ж}_2\left(\dfrac{\text{ʝ}_2}{1-\text{ʝ}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}, \\ \dfrac{1}{1+\left\{\text{Ж}_2\left(\dfrac{1-\text{œ}_2}{\text{œ}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},1-\dfrac{1}{1+\left\{\text{Ж}_2\left(\dfrac{\text{ў}_2}{1-\text{ў}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}} \end{array}\right)$$

When we take $n=2$ under Dombi AOs of Cir-IFVs, we get the following result:

$$Cir-IFDPWA(\tau_1,\tau_2)=\text{Ж}_1\tau_1\oplus\text{Ж}_2\tau_2$$

$$=\left(\begin{array}{c} 1-\dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{\text{ʝ}_1}{1-\text{ʝ}_1}\right)^{\text{€}}+\text{Ж}_2\left(\dfrac{\text{ʝ}_2}{1-\text{ʝ}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}, \\ \dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{1-\text{œ}_1}{\text{œ}_1}\right)^{\text{€}}+\text{Ж}_2\left(\dfrac{1-\text{œ}_2}{\text{œ}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},1-\dfrac{1}{1+\left\{\text{Ж}_1\left(\dfrac{\text{ў}_1}{1-\text{ў}_1}\right)^{\text{€}}+\text{Ж}_2\left(\dfrac{\text{ў}_2}{1-\text{ў}_2}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}} \end{array}\right)$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\dfrac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\dfrac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\dfrac{\text{y̆}_i}{1 - \text{y̆}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{pmatrix}$$

So, the outcomes are actual for $n = 2$.

Consider that the provided outcomes are actual for $n = s$.

Then, we get:

$$Cir - IFDPWA\,(\tau_1, \tau_2, \ldots\ldots, \tau_s) = \bigoplus_{i=1}^{s} \text{Ж}_i \tau_i$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{\text{y̆}_i}{1 - \text{y̆}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{pmatrix}$$

Now, for $n = s + 1$ then, we have

$$Cir - IFPDWA\,(\tau_1, \tau_2, \ldots\ldots, \tau_s, \tau_{s+1}) = \bigoplus_{i=1}^{s+1} \text{Ж}_i \tau_i$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} \text{Ж}_i \left(\dfrac{\text{y̆}_i}{1 - \text{y̆}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{pmatrix}$$

$$\oplus \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \text{Ж}_{s+1} \left(\dfrac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \text{Ж}_{s+1} \left(\dfrac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \text{Ж}_{s+1} \left(\dfrac{\text{y̆}_i}{1 - \text{y̆}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} \text{Ж}_i \left(\dfrac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} \text{Ж}_i \left(\dfrac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} \text{Ж}_i \left(\dfrac{\text{y̆}_i}{1 - \text{y̆}_i} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{pmatrix}$$

Therefore, the results are true for $n = s$ and $n = s + 1$. Thus, it is valid for all absolute values.

The property of idempotency guarantees that the same operation gives us consistent results when run on the same input, as far as aggregation is concerned. Monotonicity ensures that the growth of input values will not reduce the outcome, maintaining logical order. Boundedness limits the range that the output can vary (typically $[0,1]$), so that results are interpretable and the output is realistic. These properties are necessary for the reliability and validity of fuzzy set theory's operation.

Theorem 2. (Idempotency property) If $\tau_i = (ȷ_i, œ_i, ÿ_i)(i = 1,2,\ldots\ldots,n)$ are a collection of Cir-IFVs and all are identical, i.e., $\tau_i = \tau$ for all i, where $\tau = (ȷ, œ, ÿ)$. Then:

$$Cir - IFDPWA\ (\tau_1, \tau_2, \ldots\ldots, \tau_n) = \tau$$

Proof. We have

$$Cir - IFDPWA(\tau_1, \tau_2, \ldots\ldots, \tau_n) = \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{ȷ_i}{1 - ȷ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}, \\ \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{1 - œ_i}{œ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}, 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{ÿ_i}{1 - ÿ_i}\right)^{₢}\right\}^{\frac{1}{₢}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left(\dfrac{ȷ}{1 - ȷ}\right)\{\sum_{i=1}^{n} Ж_i\}^{\frac{1}{₢}}}, \\ \dfrac{1}{1 + \left(\dfrac{1 - œ}{œ}\right)\{\sum_{i=1}^{n} Ж_i\}^{\frac{1}{₢}}}, 1 - \dfrac{1}{1 + \left(\dfrac{ÿ}{1 - ÿ}\right)\{\sum_{i=1}^{n} Ж_i\}^{\frac{1}{₢}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left(\dfrac{ȷ}{1 - ȷ}\right)}, \\ \dfrac{1}{1 + \left(\dfrac{1 - œ}{œ}\right)}, 1 - \dfrac{1}{1 + \left(\dfrac{ÿ}{1 - ÿ}\right)} \end{pmatrix}$$

$$= (ȷ, œ, ÿ) = \tau$$

Theorem 3. (Boundedness property) Let $\tau_i = (ȷ_i, œ_i, ÿ_i), (i = 1,2,\ldots\ldots,n)$ be a number of Cir-IFVs. Let $\tau^+ = max\{\tau_1, \tau_2, \ldots\ldots, \tau_n\}$ and $\tau^- = min\{\tau_1, \tau_2, \ldots\ldots, \tau_n\}$. Then:

$$\tau^- \leq\ Cir - IFDPWA(\tau_1, \tau_2, \ldots\ldots, \tau_n) \leq \tau^+.$$

Proof. Consider $\tau_i = (ȷ_i, œ_i, ÿ_i)(i = 1,2,\ldots\ldots,n)$ be a number of Cir-IFVs. Assume $\tau^+ = max\{\tau_1, \tau_2, \ldots\ldots, \tau_n\} = (ȷ^+, œ^+, ÿ^+)$ and $\tau^- = min\{\tau_1, \tau_2, \ldots\ldots, \tau_n\} = (ȷ^-, œ^-, ÿ^-)$where $ȷ^- = min_k\{ȷ_k\}$, $œ^- = max_k\{œ_k\}$, $ÿ^- = min_k\{ÿ_k\}$, $ȷ^+ = max_k\{ȷ_k\}$, $œ^+ = min_k\{œ_k\}$, and $ÿ^+ = max_k\{ÿ_k\}$.

Now:

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ʝ}^-}{1-\text{ʝ}^-}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ʝ}_i}{1-\text{ʝ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ʝ}^+}{1-\text{ʝ}^+}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},$$

$$\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{œ}^+}{\text{œ}^+}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq \frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{œ}_i}{\text{œ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq \frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{œ}^-}{\text{œ}^-}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},$$

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ў}^-}{1-\text{ў}^-}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ў}_i}{1-\text{ў}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ў}^+}{1-\text{ў}^+}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}$$

Therefore:

$$\tau^-\leq Cir-IFDPWA(\tau_1,\tau_2,\ldots\ldots,\tau_n)\leq\tau^+$$

Theorem 4. (Monotonicity property) Let τ_i and $\tau_i{}'(i=1,2,\ldots.,n)$ be two groups of Cir-IFVs, $\tau_i\leq\tau_i{}'$ for all i; then:

$$Cir-IFDPWA(\tau_1,\tau_2,\ldots\ldots,\tau_n)\leq\ Cir-IFDPWA(\tau_1',\tau_2',\ldots\ldots,\tau_n').$$

Proof:

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ʝ}_i}{1-\text{ʝ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ʝ}_i{}'}{1-\text{ʝ}_i{}'}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}$$

$$\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{œ}_i}{\text{œ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq \frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{œ}_i}{\text{œ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}$$

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ў}_i}{1-\text{ў}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\leq 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{ў}_i}{1-\text{ў}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}$$

Theorem 5. Let $\tau_i=(\text{ʝ}_i,\text{œ}_i,\text{ў}_i),(i=1,2,\ldots\ldots,n)$, be a value of Cir-IFVs, and therefore, the combined outcomes of them, when applied to Cir-IFDWG, are also Cir-IFVs.

$$Cir-IFDPWG\ (\tau_1,\tau_2,\ldots\ldots,\tau_n)=\otimes_{i=1}^{n}(\tau_i)^{\text{Ж}_i}$$

$$=\left(\begin{array}{c}\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{ʝ}_i}{\text{ʝ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},\\ 1-\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{\text{œ}_i}{1-\text{œ}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}},\frac{1}{1+\left\{\sum_{i=1}^{n}\text{Ж}_i\left(\frac{1-\text{ў}_i}{\text{ў}_i}\right)^{\text{€}}\right\}^{\frac{1}{\text{€}}}}\end{array}\right)\qquad(6)$$

where $\text{Ж}_i = \frac{\Delta_i}{\sum_{i=1}^{n} \Delta_i}$, $\Delta_i = \prod_{i=1}^{n} \phi(\tau)$ and $\Delta_1 = 1$ form the weight vector (WV) of $\tau_i (i = 1,2, \ldots, n)$, $\text{Ж}_i > 0$, and $\sum_{i=1}^{n} \text{Ж}_i = 1$.

Proof. We prove Theorem 5 by the technique of induction.

Consider two Cir-IFVs

$$\tau_1 = \begin{pmatrix} \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{1 - \text{ʝ}_1}{\text{ʝ}_1} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, \\ 1 - \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{\text{œ}_1}{1 - \text{œ}_1} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{1 - \text{ў}_1}{\text{ў}_1} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}} \end{pmatrix}$$

$$\tau_2 = \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \text{Ж}_2 \left(\frac{\text{ʝ}_2}{1 - \text{ʝ}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, \\ \dfrac{1}{1 + \left\{ \text{Ж}_2 \left(\frac{1 - \text{œ}_2}{\text{œ}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, 1 - \dfrac{1}{1 + \left\{ \text{Ж}_2 \left(\frac{\text{ў}_2}{1 - \text{ў}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}} \end{pmatrix}$$

When we take $n = 2$ under Dombi AOs of Cir-IFVs, we get the following outcomes:

$$Cir - IFDPWG(\tau_1, \tau_2) = {\tau_1}^{\text{Ж}_1} \otimes {\tau_2}^{\text{Ж}_2}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{\text{ʝ}_1}{1 - \text{ʝ}_1} \right)^{\text{₢}} + \text{Ж}_2 \left(\frac{\text{ʝ}_2}{1 - \text{ʝ}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, \\ \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{1 - \text{œ}_1}{\text{œ}_1} \right)^{\text{₢}} + \text{Ж}_2 \left(\frac{1 - \text{œ}_2}{\text{œ}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, 1 - \dfrac{1}{1 + \left\{ \text{Ж}_1 \left(\frac{\text{ў}_1}{1 - \text{ў}_1} \right)^{\text{₢}} + \text{Ж}_2 \left(\frac{\text{ў}_2}{1 - \text{ў}_2} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}} \end{pmatrix}$$

$$= \begin{pmatrix} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\frac{\text{ʝ}_i}{1 - \text{ʝ}_i} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\frac{1 - \text{œ}_i}{\text{œ}_i} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{2} \text{Ж}_i \left(\frac{\text{ў}_i}{1 - \text{ў}_i} \right)^{\text{₢}} \right\}^{\frac{1}{\text{₢}}}} \end{pmatrix}$$

So, the outcomes are valid for $n = 2$.

Suppose that the provided outcomes are valid for $n = s$.

So, we get:

$$Cir - IFDPWG\ (\tau_1, \tau_2, \ldots\ldots, \tau_s) = \otimes_{i=1}^{s} \tau_i^{Ж_i}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{ȷ_i}{1 - ȷ_i} \right)^{€} \right\}^{\frac{1}{€}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{1 - œ_i}{œ_i} \right)^{€} \right\}^{\frac{1}{€}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{ў_i}{1 - ў_i} \right)^{€} \right\}^{\frac{1}{€}}} \end{array} \right)$$

Now, for $n = s + 1$ then, we have

$$Cir - IFDPWG\ (\tau_1, \tau_2, \ldots\ldots, \tau_s, \tau_{s+1}) = \otimes_{i=1}^{s+1} \tau_i^{Ж_i}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{ȷ_i}{1 - ȷ_i} \right)^{€} \right\}^{\frac{1}{€}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{1 - œ_i}{œ_i} \right)^{€} \right\}^{\frac{1}{€}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s} Ж_i \left(\dfrac{ў_i}{1 - ў_i} \right)^{€} \right\}^{\frac{1}{€}}} \end{array} \right)$$

$$\otimes \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{ Ж_{s+1} \left(\dfrac{ȷ_i}{1 - ȷ_i} \right)^{€} \right\}^{\frac{1}{€}}}, \\ \dfrac{1}{1 + \left\{ Ж_{s+1} \left(\dfrac{1 - œ_i}{œ_i} \right)^{€} \right\}^{\frac{1}{€}}}, 1 - \dfrac{1}{1 + \left\{ Ж_{s+1} \left(\dfrac{ў_i}{1 - ў_i} \right)^{€} \right\}^{\frac{1}{€}}} \end{array} \right)$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} Ж_i \left(\dfrac{ȷ_i}{1 - ȷ_i} \right)^{€} \right\}^{\frac{1}{€}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} Ж_i \left(\dfrac{1 - œ_i}{œ_i} \right)^{€} \right\}^{\frac{1}{€}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{s+1} Ж_i \left(\dfrac{ў_i}{1 - ў_i} \right)^{€} \right\}^{\frac{1}{€}}} \end{array} \right)$$

Therefore, the results are true for $n = s$ and $n = s + 1$. Therefore, it is valid for all natural numbers N.

Theorem 6. (Idempotency property) If $\tau = (ȷ_i, œ_i, ў_i)(i = 1, 2, \ldots\ldots, n)$ are a collection of Cir-IFVs, and all are identical, i.e., $\tau_i = \tau$, where $\tau = (ȷ, œ, ў)$. Then:

$$Cir - IFDPWG\ (\tau_1, \tau_2, \ldots\ldots, \tau_n) = \tau$$

Proof. We have

$$Cir-IFDPWG(\tau_1,\tau_2,\ldots\ldots,\tau_n)=\left(\begin{array}{c}1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ȷ_i}{1-ȷ_i}\right)^{₢}\right\}^{\frac{1}{₢}}},\\ \frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{1-œ_i}{œ_i}\right)^{₢}\right\}^{\frac{1}{₢}}},1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ÿ_i}{1-ÿ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}\end{array}\right)$$

$$=\left(\begin{array}{c}1-\frac{1}{\left(\frac{ȷ}{1-ȷ}\right)\{\sum_{i=1}^{n}Ж_i\}^{\frac{1}{₢}}},\\ \frac{1}{\left(\frac{1-œ}{œ}\right)\{\sum_{i=1}^{n}Ж_i\}^{\frac{1}{₢}}},1-\frac{1}{\left(\frac{ÿ}{1-ÿ}\right)\{\sum_{i=1}^{n}Ж_i\}^{\frac{1}{₢}}}\end{array}\right)$$

$$=\left(\begin{array}{c}1-\frac{1}{1+\left(\frac{ȷ}{1-ȷ}\right)},\\ \frac{1}{1+\left(\frac{1-œ}{œ}\right)},1-\frac{1}{1+\left(\frac{ÿ}{1-ÿ}\right)}\end{array}\right)$$

$$=(ȷ,œ,ÿ)=\tau$$

Theorem 7. (Boundedness property) Let $\tau_i=(ȷ_i,œ_i,ÿ_i),(i=1,2,\ldots\ldots,n)$ be a number of Cir-IFVs. Let $\tau^+=max\{\tau_1,\tau_2,\ldots\ldots,\tau_n\}$ and $\tau^-=min\{\tau_1,\tau_2,\ldots\ldots,\tau_n\}$. Then:

$$\tau^-\leq\ Cir-IFDPWG(\tau_1,\tau_2,\ldots\ldots,\tau_n)\leq\tau^+.$$

Proof. Consider $\tau_i=(ȷ_i,œ_i,ÿ_i)(i=1,2,\ldots\ldots,n)$ be a number of Cir-IFVs. Assume $\tau^+=max\{\tau_1,\tau_2,\ldots\ldots,\tau_n\}=(ȷ^+,œ^+,ÿ^+)$ and $\tau^-=min\{\tau_1,\tau_2,\ldots\ldots,\tau_n\}=(ȷ^-,œ^-,ÿ^-)$where $ȷ^-=min_k\{ȷ_k\}$, $œ^-=max_k\{œ_k\}$, $ÿ^-=min_k\{ÿ_k\}$, $ȷ^+=max_k\{ȷ_k\}$, $œ^+=min_k\{œ_k\}$, and $ÿ^+=max_k\{ÿ_k\}$.

Now:

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ȷ^-}{1-ȷ^-}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ȷ_i}{1-ȷ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ȷ^+}{1-ȷ^+}\right)^{₢}\right\}^{\frac{1}{₢}}},$$

$$\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{1-œ^+}{œ^+}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{1-œ_i}{œ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{1-œ^-}{œ^-}\right)^{₢}\right\}^{\frac{1}{₢}}},$$

$$1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ÿ^-}{1-ÿ^-}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ÿ_i}{1-ÿ_i}\right)^{₢}\right\}^{\frac{1}{₢}}}\leq1-\frac{1}{1+\left\{\sum_{i=1}^{n}Ж_i\left(\frac{ÿ^+}{1-ÿ^+}\right)^{₢}\right\}^{\frac{1}{₢}}}$$

So:

$$\tau^-\leq\mathrm{Cir-IFDPWG}\left(\tau_{k_1},\tau_{k_2},\ldots\ldots,\tau_{k_n}\right)\leq\tau^+$$

Theorem 8. (Monotonicity property) Let τ_i and $\tau_i{}'(i = 1, 2, \ldots, n)$ be two groups of Cir-IFVs, $\tau_i \leq \tau_i{}'$ for all i; then:

$$Cir - IFDPWG\left(\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_n}\right) \leq Cir - IFDPWG\left(\tau'_{k_1}, \tau'_{k_2}, \ldots, \tau'_{k_n}\right)$$

Proof: We can use the steps of Theorem 4 to prove this theorem similarly.

4 | PROPOSED MADM ALGORITHM

In this part, we propose a MADM technique that utilizes the Cir-IFDPWA and Cir-IFDPWG operators, where attributes are Cir-IFVs, and the weights of attributes are fundamental values. Therefore, the MADM problem is used to develop effective valuation methods, leveraging novelty monetization through circular information flow. For the MADM issue, consider $ҧ_i = \{ҧ_1, ҧ_2, \ldots, ҧ_{₢}\}$ be a distinct group of ₢ alternatives and $Q = \{Q_1, Q_2, \ldots, Q_n\}$ be the attributes to assess the best alternative. Consider $Ж_i = \{Ж_1, Ж_2, \ldots Ж_n\}$ be the WV of attributes, where $Ж_k(k = 1, 2, 3, \ldots, n)$ are all real values, such as $Ж_k > 0$ and $\sum_{k=1}^{n} Ж_k = 1$. Assume that $\mathbb{M} = (\tau_{ij})_{₢\times n} = (ʝ_{ij}, œ_{ij}, ÿ_{ij})_{₢\times n}$ is the Cir-IFVs decision matrix, where τ_{ij} is the feasible value for which the alternative J_i fulfills the attribute Q_j where $0 \leq ʝ_{ij} + œ_{ij} \leq 1$ and $ÿ_{ij} \in (0, \sqrt{2})$.

We attempt to address MADM challenges with the Cir-IFVs information by applying the proposed Cir-IFDPWA and Cir-IFDPWG operators. The graphically represented proposed algorithm based on Cir-IFS is displayed in Figure 2.

Step 1. Construct a Cir-IFVs decision matrix $\mathbb{M} = (\tau_{ij})_{₢\times n} = (ʝ_{ij}, œ_{ij}, ÿ_{ij})_{₢\times n}$.

Step 2. If the selected attributes are of two types, such as cost type and beneficial type, then we need to normalize the Cir-IFVs decision matrix $\mathbb{M} = (\tau_{ij})_{₢\times n} = (ʝ_{ij}, œ_{ij}, ÿ_{ij})_{₢\times n}$ into $M' = (\tau'_{ij})_{₢\times n} = (ʝ'_{ij}, œ'_{ij}, ÿ'_{ij})_{₢\times n}$ by the following equation

$$\tau'_{ij} = \begin{cases} (ʝ_{ij}, œ_{ij}, ÿ_{ij}), & if\ Q_j \text{ is a benefit attribute} \\ (ʝ_{ij}, œ_{ij}, ÿ_{ij}), & if\ Q_j \text{ is a cost attribute} \end{cases}$$

Step 3. Calculate the summation of the material σ_k of the alternative J_i Using the given l equation:

$$Cir - IFDPWA\left(\tau_{k_1}, \tau_{k_2}, \ldots, \tau_{k_n}\right) = \bigoplus_{i=1}^{n} Ж_i \tau_{ki}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{ʝ_{ki}}{1 - ʝ_{ki}}\right)^{₢}\right\}^{\frac{1}{₢}}}, \\ \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{1 - œ_{ki}}{œ_{ki}}\right)^{₢}\right\}^{\frac{1}{₢}}}, 1 - \dfrac{1}{1 + \left\{\sum_{i=1}^{n} Ж_i \left(\dfrac{ÿ_{ki,}}{1 - ÿ_{ki,}}\right)^{₢}\right\}^{\frac{1}{₢}}} \end{array} \right)$$

And

$$Cir - IFDPWG(\tau_{k_1}, \tau_{k_2}, \ldots\ldots, \tau_{k_n}) = \otimes_{i=1}^{n} (\tau_{ki})^{\text{Ж}_i}$$

$$= \left(\begin{array}{c} 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{n} \text{Ж}_i \left(\dfrac{1 - \text{ȷ}_{ki}}{\text{ȷ}_{ki}} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, \\ \dfrac{1}{1 + \left\{ \sum_{i=1}^{n} \text{Ж}_i \left(\dfrac{\text{œ}_{ki}}{1 - \text{œ}_{ki}} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}}, 1 - \dfrac{1}{1 + \left\{ \sum_{i=1}^{n} \text{Ж}_i \left(\dfrac{1 - \breve{y}_{ki,}}{\breve{y}_{ki,}} \right)^{\text{Ꞓ}} \right\}^{\frac{1}{\text{Ꞓ}}}} \end{array} \right).$$

Step 4. Compute the score function $\phi(\tau)$ for each alternative J_l by Eq.

$$\phi(\tau) = (\text{ȷ}_\tau - \text{œ}_\tau) \times \breve{y}_\tau$$

Step 5. Evaluate all alternatives to determine the optimal decision among them.

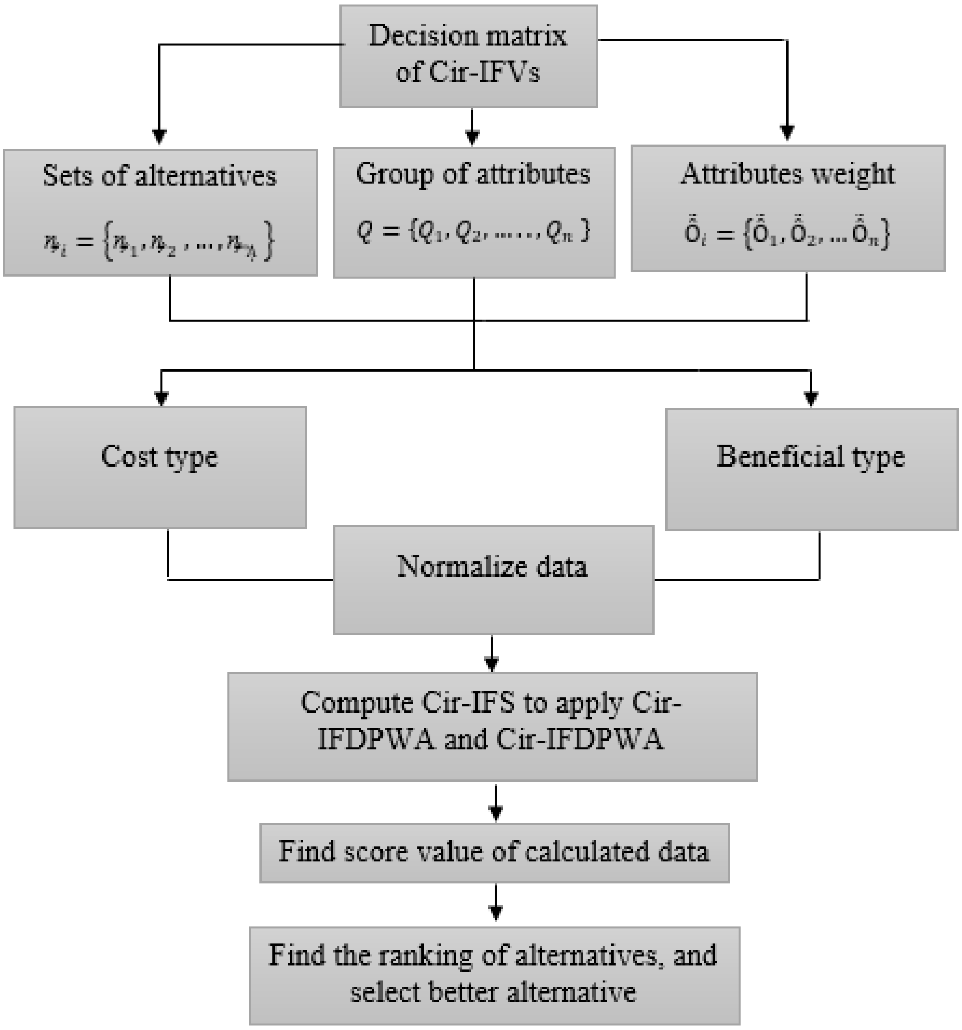

Figure 2. Algorithm of MADM based on Cir-IFDPWA and Cir-IFDPWG operators.

We observed from the above figure that, first of all, we collect information using the Cir-IFVs by defining the set of alternatives and attributes in the form of a decision matrix. Then we have applied the developed theory of Cir-IFDPWA and Cir-IFDPWG operators. We apply the score function formula to evaluate the best result. Finally, we ranked the alternatives to select the best one.

5 | CASE STUDY

A social media platform functions as a valuable online service that facilitates the creation and sharing of content, promotes interaction among users, and nurtures the development of virtual communities. These platforms will provide an opportunity to improve communication, marketing, and branding methods, as well as overall networking in agriculture, business, and education. Unknowingly, businesses, especially agricultural businesses, can effectively promote their operations by utilizing these tools effectively, as they help establish strong business relationships with customers and partners, which in turn create opportunities for growth and collaboration.

Social media platforms are vital in today's digital world. They represent more than a means of communication with other people; they can also become a significant part of your business. There are many advantages these platforms offer, and a few attributes should be considered to understand their significance:

i. **Increased visibility:** With social media, businesses have a universe of possibilities to connect with a vast audience. Today's companies can achieve significant success without wasting their resources on traditional advertising methods to reach local, national, or international markets.

ii. **Customer interaction and engagement**: The power of social media enables absolute and expressive interaction with customers. With honest time feedback from customers and good communication, we can work more effectively. This strengthens the relationship between businesses and the audience.

iii. **Cost-effective marketing**: Social media advertising is increasing and becoming more cost-effective for reaching people than traditional media, such as TV, radio, or print. Technological advances will, most likely, give rise to new, more complex approaches to marketing, such as targeted advertising, sponsored posts, and more concerted initiatives for organic engagement. Armed with this information, businesses can create more effective engagement with their consumer base and run highly effective campaigns.

iv. **Data-driven decisions:** Social media offers numerous benefits due to its abundance of helpful information. By utilizing these platforms, businesses can gain access to powerful tools that provide real-time insights into customer actions and interests. By analyzing this data, organizations can refine their strategies to better address the needs of their audience.

There are many social media platforms in the modern age to help local businesses enter the international market. They enable new companies to increase visibility, foster customer interaction and engagement, implement cost-effective marketing strategies, and make data-driven decisions. Under such considerations, one of the tasks which are not easy to deal with is to find the most appropriate alternative (Facebook, YouTube, Instagram, Telegram) to expand business promotion in both local and international markets. Figure 3 shows the various platforms for social media marketing.

a. Facebook: Facebook enables users to connect, share information, and communicate in various ways, including through posts, chats, and live broadcasts. It works well for reaching out to specific neighborhoods and for advertising.
b. YouTube: Companies can put tutorials, product demos, and promotional material on YouTube. This platform is ideal for creating in-depth videos that can captivate audiences worldwide through compelling storytelling.
c. Instagram: Since Instagram relies on pictures and short clips, it is widely used by brands and influencers to reach younger customers.
d. Telegram: Telegram suggests a safe, cloud-based messaging program with robust privacy features and high-speed communication. It helps large groups, channels, and bots, making it ideal for broadcasting content and engaging diverse audiences.

Figure 3. Different platforms for social media marketing

5.1 | Numerical Example

This part applies Cir-IFDPWA and Cir-IFDPWG aggregation operations to handle a genuine social media platform selection dilemma. The following part describes the issue at hand.

A successful online presence requires an ideal platform, as platforms serve as the fundamental source of success. Successful digital marketing strategies depend on selecting the right social media platform. The research evaluates a case that determines social media platforms from various options, as social media serves as a fundamental element for brand engagement and online communication. All major social media networks continually strive to enhance user experiences as competition between them intensifies daily in digital markets worldwide. The MADM process allows the incorporation of expert opinion when there are uncertainties to address the issue of selecting the best social media platform.

5.2 | Implementation

Let's suppose we are selecting the most appropriate social media platform from four alternatives, like $ᵰ_i = \ (i = 1,2,\dots,4)$ These are the attribute keys for the selected alternatives. Such as Q_1 is the increased visibility, Q_2 is the customer interaction and engagement Q_3 is cost-effective marketing and Q_4 Data-driven decisions. The attribute weight $£ = (0.58, 0.23, 0.14, 0.05)$ are distributed by Decision Matrix DMs. The DMs evaluate four social media platforms $ᵰ_i = \ (i = 1,2,\dots,4)$ The use of ambiguity with Cir-IFS based on four attributes is presented in Table 1.

Step 1. Create a Cir-IFVs decision matrix $\mathbb{M} = (\tau_{ij})_{₲\times n} = (ȷ_{ij,}\ œ_{ij}, ÿ_{ij})_{₲\times n}$. Decision matrix $\mathbb{M} = (\tau_{ij})_{₲\times n}$ in the form of Cir-IFVs provided in Table 1.

Table 1. Decision matrix

$ᵰ_i$	Q_1	Q_2	Q_3	Q_4
$ᵰ_1$	$(0.3, 0.1, 0.2)$	$(0.4, 0.2, 0.3)$	$(0.3, 0.2, 0.5)$	$(0.2, 0.4, 0.2)$
$ᵰ_2$	$(0.2, 0.4, 0.1)$	$(0.3, 0.4, 0.1)$	$(0.2, 0.3, 0.1)$	$(0.4, 0.3, 0.1)$
$ᵰ_3$	$(0.5, 0.3, 0.1)$	$(0.2, 0.5, 0.1)$	$(0.3, 0.4, 0.2)$	$(0.5, 0.1, 0.3)$
$ᵰ_4$	$(0.1, 0.5, 0.3)$	$(0.2, 0.4, 0.1)$	$(0.2, 0.3, 0.4)$	$(0.3, 0.4, 0.1)$

Step 2. The selected attributes in the beneficial type, then no need to normalize the Cir-IFVs decision matrix $\mathbb{M} = (\tau_{ij})_{₲\times n} = (ȷ_{ij,}\ œ_{ij}, ÿ_{ij})_{₲\times n}$

Step 3. Estimate the collective material σ_k of the alternative J_i Provided in Table 1 using proposed AOs as Cir-IFDPWA and Cir-IFDPWG operators, and the aggregated outcomes are presented in Table 2.

Table 2. Aggregation findings by using the proposed Cir-IFDPWA and Cir-IFDPWG.

$ꞥ_i$	Cir-IFDPWA	Cir-IFDPWG
$ꞥ_1$	$(0.3275, 0.1210, 0.3188)$	$(0.3031, 0.1893, 0.2277)$
$ꞥ_2$	$(0.4493, 0.3727, 0.0998)$	$(0.2180, 0.3857, 0.1002)$
$ꞥ_3$	$(0.4493, 0.2690, 0.1414)$	$(0.3076, 0.3872, 0.1081)$
$ꞥ_4$	$(0.1643, 0.4165, 0.2932)$	$(0.1206, 0.4604, 0.1645)$

Step 4. Compute the score function $\phi(\tau)$ for each alternative J_l by using Table 2. The analyzed outcomes are provided in Table 3, and the graphical representation of the outcomes is shown in Figure 4.

Table 3. Scores of aggregated information.

$ꞥ_i$	Cir-IFDPWA	Cir-IFDPWG
$ꞥ_1$	0.0658	0.0259
$ꞥ_2$	-0.0126	-0.0168
$ꞥ_3$	0.0255	-0.0086
$ꞥ_4$	-0.0740	-0.0559

Step 5. In this step, the score function value is used to estimate the ranking, and then all alternatives are evaluated to determine the optimal decision among them. The ranking of selected options is in Table 4, where we noticed that the best alternative is $ꞥ_1$ by using the Cir-IFDPWA and Cir-IFDPWG operators.

Table 4. Scores of aggregated information.

Cir-IFDPWA	$ꞥ_1 > ꞥ_3 > ꞥ_2 > ꞥ_4$
Cir-IFDPWG	$ꞥ_1 > ꞥ_3 > ꞥ_2 > ꞥ_4$

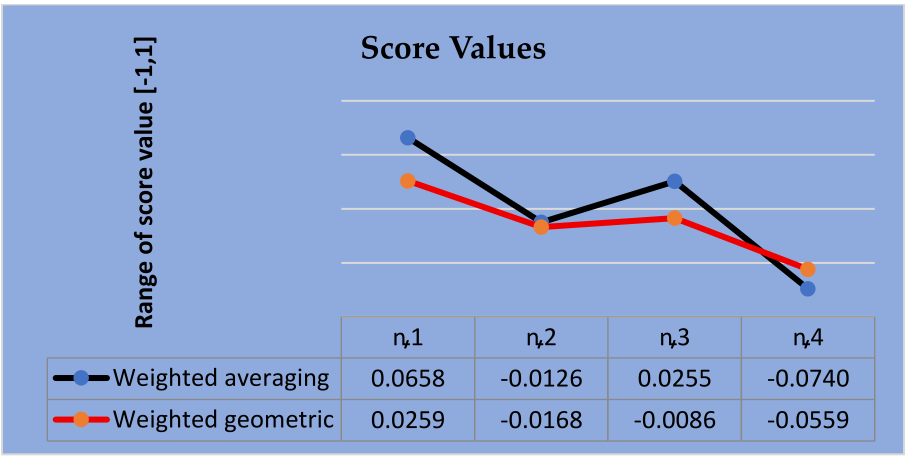

Figure 4. Geometrical representation of score values.

In Figure 4, the red line represents the score value obtained using the Cir-IFDPWG operator, and the best alternative is $ɳ_1$, and the black line represents the score value obtained using the Cir-IFDPWA operator, and the best option is $ɳ_1$. We notice in Table 4 and Figure 4 that the best alternative is $ɳ_1$ when we use the Cir-IFDPWA and Cir-IFDPWG operators.

6 | COMPARATIVE ANALYSIS

Now, we will assess the performance of the established method with the existing AOs, circular intuitionistic fuzzy weighted averaging (Cir-IFWA) and circular intuitionistic fuzzy weighted geometric (Cir-IFWG) as proposed by [27], and circular intuitionistic fuzzy Hamacher weighted averaging (Cir-IFHWA) and circular intuitionistic fuzzy Hamacher weighted geometric (Cir-IFHWG) established by [28]. We will take the information from the case study and Table 1 for this comparative analysis, and the results are obtainable in Table 5. The method with intuitionistic fuzzy rough power AOs (IFRPAOs) established by [20], offered intuitionistic fuzzy Hamacher weighted averaging (IFHWA). The theory of circular intuitionistic fuzzy Heronian means (C-IFHM) provided by Guo et al. [29] and intuitionistic fuzzy Hamacher weighted geometric (IFHWG) given by Huang [30] cannot create clear outcomes due to restrictions basic in their outline. The real-life example presented in this article contains a radius that the Cir-IFS, as mentioned earlier, cannot handle.

Table 5. Show the comparative analysis of established work and existing work

	Operator	Ranking
Proposed method	Cir-IFDPWA	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
	Cir-IFDPWG	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
Chen [27]	Cir-IFWA	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
	Cir-IFWA	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
Fahmi et al. [28]	Cir-IFHWA	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
	Cir-IFHWG	$ɳ_1 > ɳ_3 > ɳ_2 > ɳ_4$
Khan et al. [20]	IFRPAOs	Fail
Guo et al. [29]	C-IFHM	Fail
Huang [30]	IFHWA	Fail
	IFHWG	Fail

Through comparative analysis, we examine the existing technique and the established method, along with their ranking outcomes, as presented in Table 5 and Figure 5. This illustration presents the geometrical representation that supports the consideration of this comparative study.

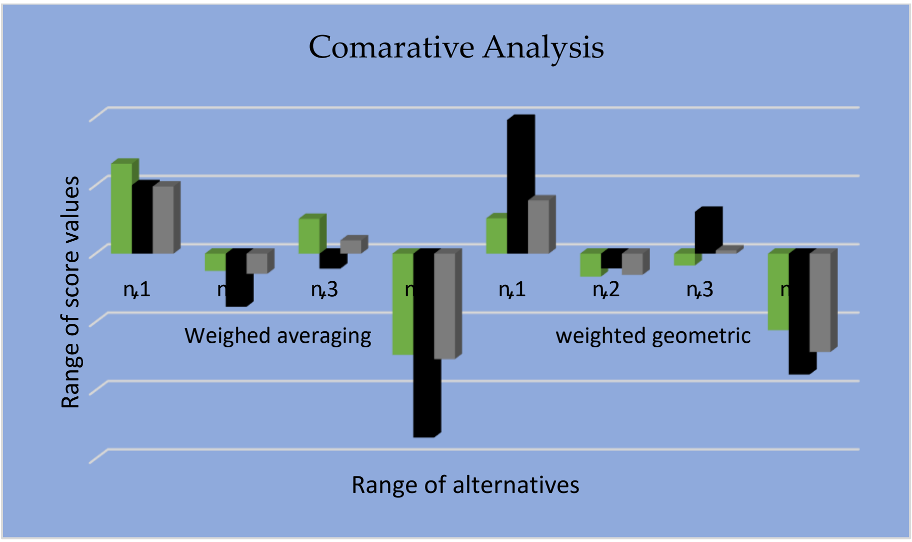

Figure 5. Graphical representation of the comparison of the established and existing AOs.

The advantages and benefits of the established idea are evident in this article, which represents a generalized form of the IFS. If we remove the radius value, then the Cir-IFS converts to the IFS, and Definition 1 for the Cir-IFS converts to the IFS. Therefore, the Cir-IFS is capable of handling the doubt and ambiguity of the framework. When we compare Cir-IFWA, Cir-IFWG, Cir-IFHWA, Cir-IFHWG operators, then the best alternative is $ɳ_1$.

There are many more real-life applications of the proposed work that exist in the modern age. In the present era, the concept of decision-making is necessary for every aspect of life. For example, if a person wants to find the best car for home use. He has many choices in the market, including Honda, Toyota, Hyundai, and Suzuki. When selecting the best car, a decision maker considers several key attributes, such as fuel efficiency, resale market value, parts availability, build quality, and more. For investigating these types of real-life problems, our proposed MADM model is a suitable tool, providing logical and accurate results.

7 | CONCLUSION

This study established a novel set of AOs, Cir-IFDPWA and Cir-IFDPWG, that combine the soulfulness of Cir-IFS with the operational flexibility of Dombi operators. Through rigorous axiomatic validation, both operators are demonstrated to satisfy key mathematical properties essential for reliable aggregation in the MADM framework. Our established operators are accustomed to real-life scenarios that connect the selection of the best social media platforms to their practical value and inherent power. The comparative study exposes that Cir-IFDPWG tends to produce more characteristic decision results, while Cir-IFDPWA suggests a smoother compromise across incompatible attributes. Both AOs form a valuable technique for decision experts dealing with vague and complex attributes.

The proposed approach is a suitable tool for investigating complex decision-making problems. But there are some limitations in the proposed theory, for example, the developed theory is not applicable for the aggregation of circular Pythagorean fuzzy set (FS) (Cir-PyFS), circular q-rung orthopair FS (Cir-q-ROFS). Our developed theory is unable to handle high-order information.

Future research could explore their application in other fuzzy frameworks. For example, [31] offered a circular Pythagorean fuzzy framework, and [32] established TOPSIS using PFS-based data to select a green supplier problem. Theory of interval valued spherical fuzzy set under Dombi operation laws discussed by [33]. We also extend our proposed theory to other FS structures, such as Cir-PyFS and Cir-q-ROFS.

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