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Foundations of a Two-Time Relativistic Framework: Lorentz Symmetry, Proper Time, and Matter Structure

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ABSTRACT

Despite the success of quantum theory, its axiomatic foundations remain conceptually opaque, relying on postulates that resist geometric or physical intuition. In this work, we present a geometric framework that unifies quantum and relativistic principles through a two-dimensional concept of time, combining chronological evolution with intrinsic phase periodicity. Building on the time phase uncertainty exclusion interference triad introduced in earlier work [1],[2]. We reformulate the Lorentz transformation to incorporate geometric constraints and derive a correction relevant for particles on collision trajectories. This yields a refined interpretation of relativistic effects such as length contraction, invariants, and proper time. Particle motion is describe using extended 5-vectors, enabling the formulation of a generalized velocity and a new Lagrangian from which relativistic energy and momentum naturally emerge. The framework further suggests a geometric origin for the structure of matter and electric charge, grounded in time phase topology. These results will not only reproduce known features of special relativity and quantum mechanics but also point toward a unified, axiomatic reformulation of quantum field theory rooted in geometric first principles.

Keywords: Quantum theory; Relativistic principles; Collision trajectories; Lagrangian.

(MSC2020) Codes: 81Q05, 83A05, 70F40, 70H03

1 | INTRODUCTION

In spite of the empirical success of quantum mechanics, its axiomatic structure including the postulates of wavefunction collapse, canonical commutation relations, and the Born rule remains conceptually opaque, lacking a direct geometric or physical interpretation. Unlike general relativity, which derives its entire formalism from spacetime geometry, quantum theory continues to rely on ad hoc assumptions that resist intuitive understanding. It gives rise to long-standing challenges such as the measurement problem, nonlocality, and the absence of a unified ontological framework [3],[4],[5]. Efforts to resolve these conceptual issues have led to the development of dynamical collapse models [3], hidden variable theories, and recent geometric approaches aimed at deriving quantum mechanics from structures on projective Hilbert space or symplectic manifolds [6],[7]. While insightful, these frameworks have yet to offer a complete, physically grounded explanation of fundamental quantum features such as interference, exclusion, and uncertainty as consequences of a deeper spacetime geometry. Leading thinkers continue to stress the disconnect between mathematical consistency and physical clarity in quantum theory [4], [8],[9], [10]. For example, while classical mechanics and relativity achieve explanatory power through geometric intuition, quantum theory remains formally effective yet foundationally unsettled [9],[10],[11],[12]. This asymmetry motivates the search for a unified geometric foundation that can render quantum principles as inevitable consequences of deeper spacetime symmetries and constraints.

In prior work [1],[2], we introduced a geometric model in which time is treated as a two-dimensional entity. This construction provides a natural setting for deriving quantum principles from first geometric principles rather than relying on postulates. In [1], we demonstrated how this two-dimensional structure reconciles the discreteness of quantum phenomena with the continuous symmetries of special relativity, offering a unified temporal framework that maintains Lorentz invariance while accommodating interference and quantization. Expanding on this foundation, reference [2] introduced the uncertainty exclusion interference triad as a consequence of constraints in time phase geometry. It was shown that Heisenberg-like uncertainty relations, Pauli exclusion, and path-based interference phenomena emerge from geometric relationships in the time phase manifold. This analysis emphasized that canonical quantum behavior normally imposed through operator algebra or abstract Hilbert space axioms can instead be derived from geometric constraints on allowable trajectories through time phase space. This reinterpretation of core quantum principles as geometric consequences offers a potential bridge between quantum theory and relativistic spacetime. However, while [1] and [2] laid the conceptual and mathematical groundwork, they did not yet incorporate a full treatment of particle dynamics, the Lorentz transformation, or energy momentum relations. The present work completes that program by embedding these dynamical and physical constructs into the time phase model, and by extending the analysis to matter, charge, spin, and general relativistic implications.

In our earlier studies, we derived quantum principles; uncertainty, exclusion, and interference; as natural outcomes of a two-dimensional time phase geometry [1],[2]. However, several essential aspects remain to be addressed before this framework can be considered a comprehensive foundation for a physical theory. First, relativistic kinematics including the Lorentz transformation, invariant intervals, and proper time has yet to be reformulated within the time phase context. Embedding these into a dual temporal structure is necessary to ensure consistency between the model and established relativistic phenomena. Second, the absence of geometric dynamical constructs, such as position and velocity extended beyond four-vector descriptions, prevents derivation of standard expressions for relativistic energy and momentum. Without a proper geometric Lagrangian built on these constructs, the model lacks the necessary tools to describe particle dynamics rigorously. Third, the origination of matter, charge, spin, and statistics has not yet been tied explicitly to geometric principles in the previous work. By connecting discrete quantum properties to features like compactness and trajectory distinguishability within time phase space, we aim to show how both bosonic and fermionic behaviors and even electric charge can emerge from topological and geometric constraints.

Finally, extending the model toward a coherent description of geometro-dynamics requires evaluating how time phase geometry interfaces with curvature, energy–momentum sources, and general relativity. Demonstrating compatibility would support the model's viability as a basis for both quantum and gravitational physics. This paper addresses these unresolved questions by: (i) reformulating Lorentz transformations and defining invariants using the time phase formalism; (ii) extending kinematic variables into five-dimensional vectors and deriving a corresponding geometric Lagrangian; (iii) demonstrating how matter and charge arise from time phase topology; and (iv) exploring the framework's implications for general relativity and field dynamics. Our approach follows the philosophy of geometric and symplectic formulations in quantum mechanics [13], [14], [15], [16] and builds on recent advances in contact geometry applied to phase-space quantization [17],[18]. It also aligns with broader initiatives aiming to unify quantum discreteness and relativistic invariance in a single geometric framework [19], [20], [21], [22].

This article develops the program outlined above by providing detailed constructions, derivations, and interpretations that firmly embed relativistic and quantum phenomena within a two-dimensional time phase geometric framework [1],[2]. The first contribution involves a reinterpretation of the Lorentz transformation by explicitly incorporating both chronological and phase components of time. This leads to a corrected form of the transformation, particularly relevant for systems involving particles on collision trajectories, and clarifies subtle aspects of simultaneity, proper time, and length contraction within the extended geometry [1],[4],[13]. Next, we construct a five-dimensional velocity formalism that extends the standard four-dimensional spacetime description. This generalization allows for a coherent treatment of motion in time phase space and provides the foundation for deriving a geometric Lagrangian, from which relativistic energy and momentum relations emerge as natural consequences. This approach draws on established symplectic methods in geometric mechanics [13],[14],[15],[16] and reinforces the view that dynamical equations can be derived from topological and geometric constraints [17].

A central result of this work is the geometric origin of exclusion and indistinguishability, which are shown to arise from topological properties of phase-space trajectories. In particular, fermionic and bosonic statistics are reinterpreted as resulting from trajectory orthogonality and compactness in the phase dimension. This insight parallels early conceptual efforts by Pauli and Feynman but is here embedded in a concrete geometric model [5], [7], [11], [14] [5,7,11,14,40,41]. Furthermore, we propose a mechanism by which electric charge can be understood as a conserved quantity associated with closed trajectories in the phase component of time. This interpretation is consistent with the structure of gauge invariance and conserved No ether charges but arises here from intrinsic geometric features of time phase evolution [18],[19],[20]. Finally, we analyze the implications of this formalism for general relativity. We suggest that spacetime curvature and energy–momentum couplings in Einstein's field equations may be interpreted within the time phase model, offering a potential pathway toward unifying quantum discreteness with geometrodynamical curvature [9], [19], [22].

The paper is structured to build progressively from conceptual foundations to technical implementations within the time phase geometric framework. Following the introduction in Section 1, Section 2 reinterprets the position of a particle in space and time, with a focus on Lorentz invariance. Subsections explore a revised interpretation of the Lorentz transformation, a derivation tailored to colliding particles, and a thought experiment that provides a novel geometric understanding of length contraction. Section 3 introduces the concept of velocity as a five-dimensional vector, extending conventional four-vector formulations by incorporating the phase component of time. In Section 4, the geometric Lagrangian is constructed from this five-vector formalism, leading to expressions for relativistic momentum and energy. Section 5 discusses the broader implications of this structure for general relativity, focusing on the compatibility of time phase geometry with gravitational curvature and field equations. Finally, Section 6 examines the emergence of matter and electric charge from topological constraints in phase space, establishing a potential link between quantum numbers and underlying geometric properties.

2| POSITION OF A PARTICLE IN SPACE AND TIME

In this section, we analyze the spatial coordinate transformation between inertial observers using the dual-temporal framework that underpins our model. Building on our previous derivation of temporal trajectories, we now derive explicit equations for spatial coordinates, confirming consistency with Einstein's Lorentz transformations under modified geometric interpretation.

2.1 | A slight change on how to look at the Lorentz Transformation

Since the trajectory of a particle in time has already been formulated within our two-dimensional time framework, we now extend this treatment to determine how spatial coordinates transform between two inertial observers using our modified coordinate system [1]. To explore this, we begin with a canonical thought experiment originally analysed by [24] and later elaborated in modern pedagogical texts such as [26-27], that illustrates the relativity of simultaneity through the emission and detection of photons on a moving train.

A train moves with constant velocity past a stationary observer, denoted O. Positioned at the midpoint of the train compartment is another observer O', who is at rest relative to the train. At the precise moment when the rear of the wagon is adjacent to observer O, two photons are emitted simultaneously from the front and rear ends of the wagon in such a way that they converge at the midpoint where O' is located. Because the detection mechanism only triggers if both photons arrive simultaneously, both observers agree on the occurrence of this event. From the perspective of observer O, however, all points on the train are receding after the moment of emission until detection. This relative motion implies that the corresponding geometric parameter is $\rho = 1$. The configuration is illustrated in Figure 1.

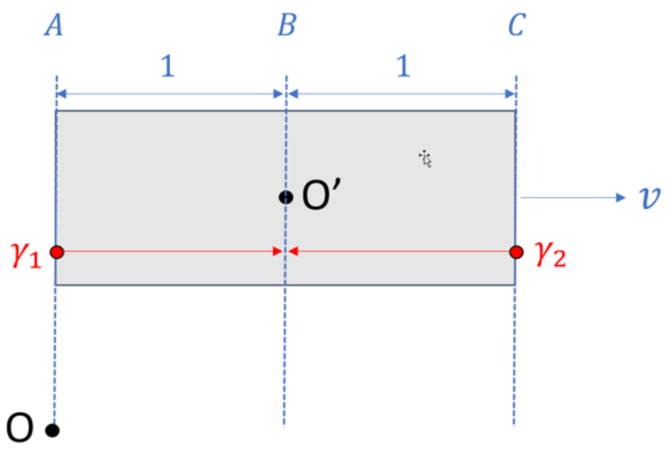


Figure 1. A thought experiment illustrating simultaneity in different inertial frames.

Observer O is stationary on the ground, while observer O' is located at the center of a train moving with velocity v. At the moment the rear of the train aligns with O, two photons ($\gamma_1$and γ_2) are emitted simultaneously from both ends of the train. The photons are arranged to meet at the midpoint, triggering a detector that both observers agree indicates simultaneous arrival.

The corresponding spacetime coordinates are [31]:

$$x' = 0, \qquad x' = 1, \qquad x' = 2,$$
$$x = vt, \qquad x = vt + 1, \qquad x = vt + 2,$$

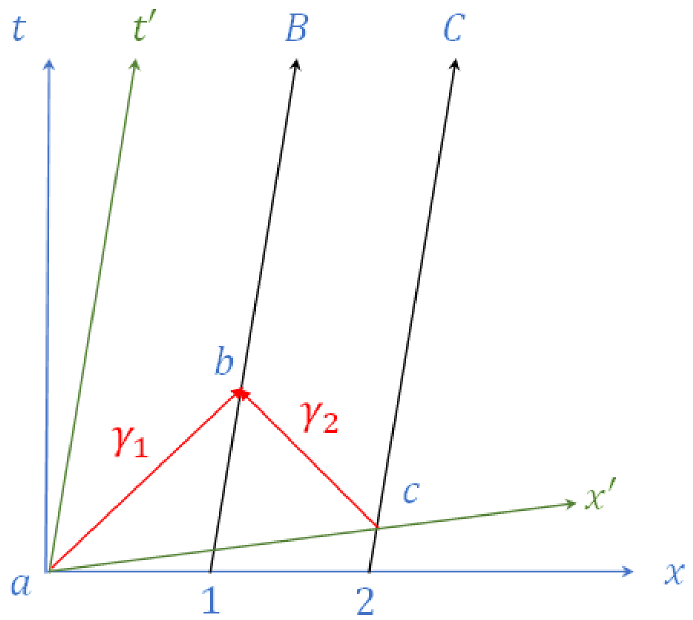


Figure 2. Spacetime diagram illustrating the event of simultaneous photon arrival in the frame of observer O.

We now derive the equation of the x'-axis, corresponding to $t' = 0$, using this geometric configuration. The line $a \to b$ represents the trajectory of a photon emitted from the rear end of the wagon to the midpoint, and since the slope of the light ray is 1 (equal to the value of ρ for γ_1 from the perspective of O) is defined by:

$$x = t.$$

The line $1 \to b$, representing the worldline of the midpoint of the wagon from the perspective of the stationary observer, is given by:

$$x = vt + 1.$$

To find the intersection point b, we solve the system

$$t_b = vt_b + 1 \to t_b = \frac{1}{1-v}, \qquad x_b = \frac{1}{1-v}.$$

Therefore, the coordinates for event b are

$$b = \left(\frac{1}{1-v}, \frac{1}{1-v}\right).$$

To determine the spatial trajectory of the photon arriving at point b from the front of the train, we consider the line $c \to b$, which represents the light ray emitted from the front of the train and received at the midpoint.

$$x = mt + k.$$

The slope of the light ray is -1 (equal to the value of ρ for γ_2 from the perspective of O):

$$x = -t + k.$$

Since the line goes through point b we can insert its coordinates to find the value of k:

$$\frac{1}{1-v} = -\frac{1}{1-v} + k \rightarrow k = \frac{2}{1-v}.$$

Therefore, the equation for the light ray is:

$$x = -t + \frac{2}{1-v}.$$

We now determine the point c where this line intersects with the worldline of the front of the wagon, which follows the trajectory:

$$x = vt + 2.$$

Equating the two expressions:

$$vt_c + 2 = -t_c + \frac{2}{1-v} \quad \rightarrow \quad t_c = \frac{2v}{1-v^2}, \quad x_c = \frac{2}{1-v^2}.$$

Therefore, the coordinates for point c are:

$$c = \left(\frac{2}{1-v^2}, \frac{2v}{1-v^2}\right).$$

The line connecting the origin to point c represents the trajectory of simultaneity in the primed frame and satisfies the relation:

$$t = vx \text{ at } t' = 0.$$

For any other constant value of t', this generalizes to:

$$t = vx + t',$$

which, in terms of the geometric sign parameter $\rho = 1$, can be written as:

$$t' = t - \rho vx, \tag{1}$$

while this derivation reproduces the standard Lorentz structure, it highlights the symmetry encoded in the parameter ρ. This distinction becomes significant when we alter the configuration such that the train moves in the opposite direction i.e., when the observer perceives the wagon approaching rather than receding.

Now consider the reverse scenario: a train moves past a stationary observer O, and at the moment the front of the wagon aligns with O, two photons are emitted from both ends of the train, simultaneously in the frame of O' at the center. The photons converge at a detector located at the midpoint, verifying simultaneous arrival. From O perspective, the detector and the rear of the train move toward him during the photons' travel, implying $\rho = -1$. The configuration is depicted in Figure 3.

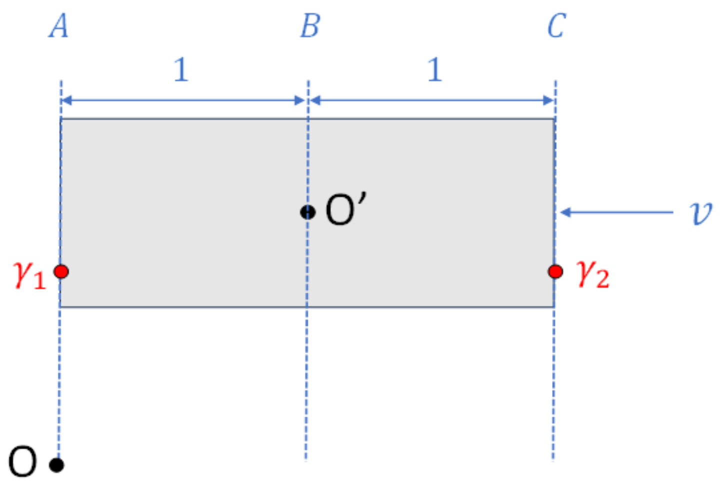


Figure 3. Modified thought experiment illustrating the case where the train is moving in the opposite direction relative to the stationary observer O.

$$x' = 0, \qquad x' = 1, \qquad x' = 2,$$
$$x = -vt, \qquad x = -vt + 1, \qquad x = -vt + 2,$$

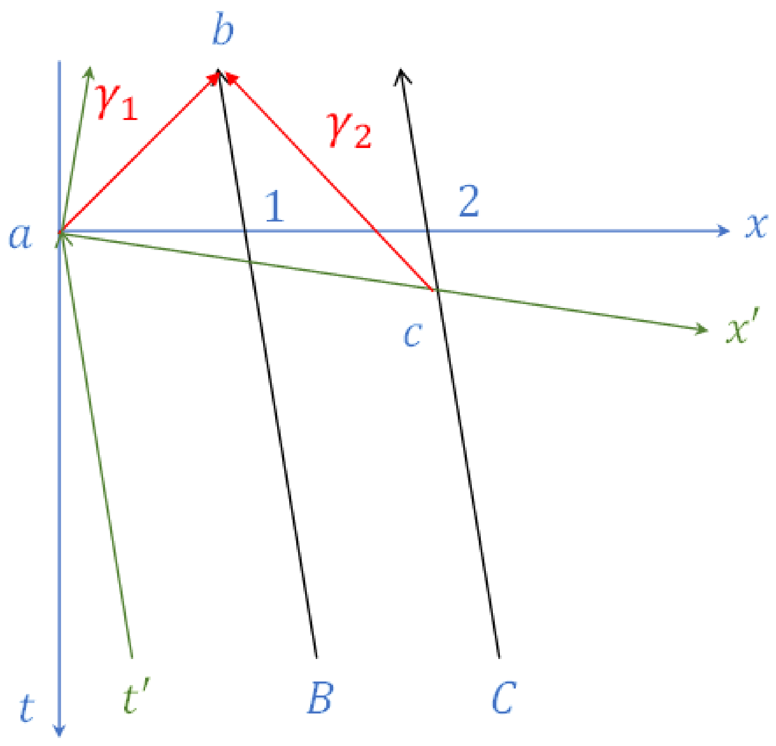


Figure 4. The corresponding spacetime diagram for this reversed configuration

We again seek the equation of the $x' -$ axis at $(t' = 0)$. The photon emitted from the rear of the wagon follows the trajectory represented by line $a \to b$, defined by (the slope being again the value of ρ for γ_1 from the perspective of O):

$$x = t.$$

The worldline of the midpoint of the wagon from observer O's perspective is ($1 \rightarrow b$):

$$x = -vt + 1.$$

Solving for the intersection point b, we set:

$$t_b = -vt_b + 1 \;\rightarrow\; t_b = \frac{1}{1+v}, \qquad x_b = \frac{1}{1+v}.$$

Therefore, the coordinates for point b are:

$$b = \left(\frac{1}{1+v}, \frac{1}{1+v}\right).$$

Find the equation for line $c \rightarrow b$, which represents the trajectory of the photon emitted from the front of the train:

$$x = mt + k.$$

The slope of the light ray is -1 (again the value of ρ for $\gamma_2$from the perspective of O):

$$x = -t + k.$$

Since the line goes through point b we can plug in its coordinates to find the value of k:

$$\frac{1}{1+v} = -\frac{1}{1+v} + k \;\rightarrow\; k = \frac{2}{1+v} \;\rightarrow\; x = -t + \frac{2}{1+v}.$$

Now consider the worldline of the front of the wagon, which is given by ($2 \rightarrow c$):

$$x = -vt + 2.$$

Setting the two expressions equal to find the intersection point c:

$$-vt_c + 2 = -t_c + \frac{2}{1+v} \;\rightarrow\; t_c = \frac{-2v}{1-v^2}, \quad x_c = \frac{2}{1-v^2}.$$

Therefore, the coordinates for point c are:

$$c = \left(\frac{2}{1-v^2}, \frac{-2v}{1-v^2}\right).$$

The line from the origin to point c has the equation:

$$t = -vx \text{ at } t' = 0.$$

which generalizes for arbitrary t' to:

$$t = -vx + t',$$

and, introducing the geometric sign parameter $\rho = -1$, we write:

$$t' = t - \rho vx\,. \qquad (2)$$

This result mirrors equation (1) and confirms the geometric symmetry of both configurations. To visualize this symmetry more generally, the spacetime diagram can be simplified as follows:

$$x' = 0, \qquad x' = 1, \qquad x' = 2,$$

$$\boldsymbol{x = \rho v t}, \qquad \boldsymbol{x = \rho v t + 1}, \qquad \boldsymbol{x = \rho v t + 2},$$

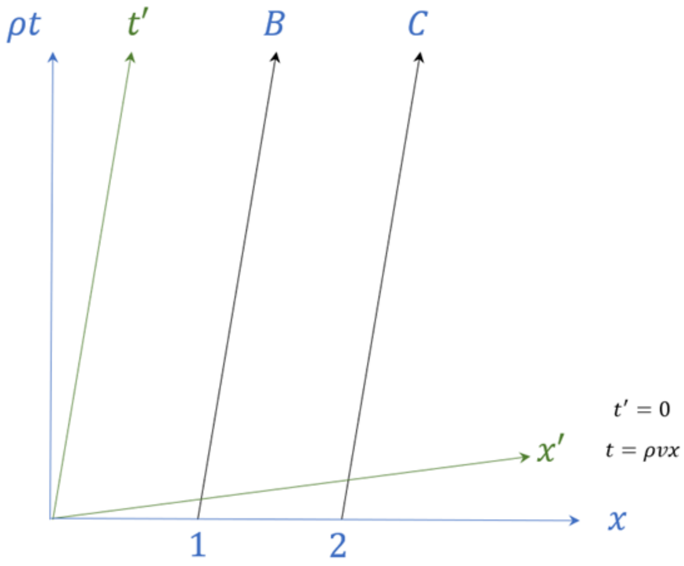
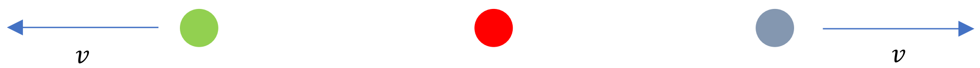


Figure 5. Unified spacetime diagram showing both receding and approaching frames within a single quadrant.

In this coordinate system both O and O' agree on the value of ρ and the value of v and x, which are all positive quantities by construction. Viewing the same configuration from the rest frame of O', we obtain the inverse relation:

$$t = t' - \rho v x'.$$

However, the symmetry in directionality invites one further generalization. The two thought experiments considered so far could equally have been drawn in mirrored configurations, involving particles moving leftward or rightward. To account for this distinction, consider a three-particle system.



In this situation both the green particle and the blue particle are moving away from the central red particle at the same speed. Therefore, from the red particle's frame, they share equal velocity magnitudes $v^{gr} = v^{br}$, motion directionality $\rho^{gr} = \rho^{br}$ and and displacements $x^{gr} = x^{br}$. Yet, these are clearly distinct particle configurations, differentiated by direction in space. To capture this distinction, we introduce an additional quantum discrete number label β, defined such that $\beta = -1$ for a particle moving to the left of the observer otherwise $\beta = 1$. In this way case $\beta^{gr} = -\beta^{br}$, allowing the diagram to incorporate both situations:

$$x' = 0, \qquad x' = \beta, \qquad x' = 2\beta,$$

$$x = \rho\beta v t, \qquad x = \rho\beta v t + \beta, \qquad x = \rho\beta v t + 2\beta,$$

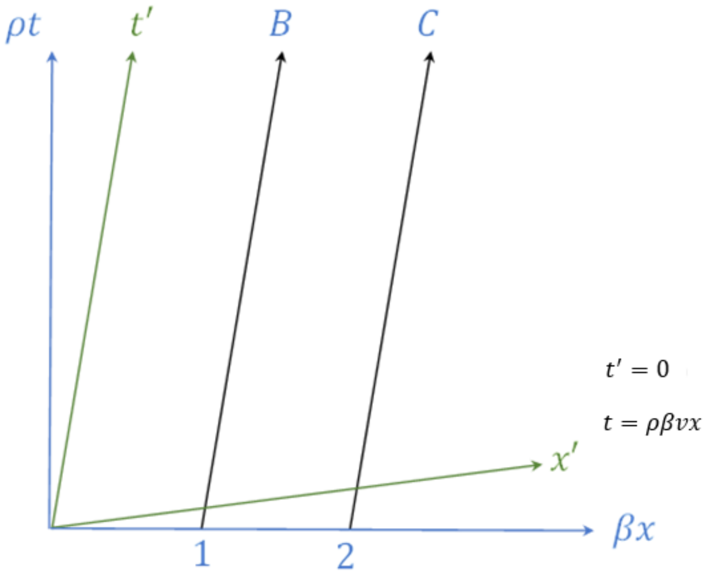


Figure 6. Generalized spacetime diagram representing symmetrical motion using both directionality (β) and relative separation (ρ).

Recognizing the difference in the real rate of time evolution between inertial frames, we now derive the time coordinate t' in the moving frame from the geometric transformation of the time component $(X^0)'$. The resulting transformation is:

$$t' = \frac{t - \rho\beta v x}{\sqrt{1 - v^2}}. \quad (3)$$

In the limit where the second dimension of time is suppressed, this reduces precisely to the Lorentz transformation for time. However, unlike the standard formulation, this expression exhibits full symmetry with respect to both velocity direction and particle separation: the sign ambiguity in the Lorentz transformation is resolved via the composite quantity $\rho\beta$, upon which both observers agree. Hence, the inverse transformation is given by:

$$t = \frac{t' - \rho\beta v x'}{\sqrt{1 - v^2}}.$$

We postpone the derivation of the Lorentz factor $\frac{1}{\sqrt{1-v^2}}$ until the next section, where we reproduce Einstein's derivation from first principles within our new geometric framework [32,33].

3 | DERIVATION OF THE LORENTZ TRANSFORMATION AND THE ERROR FOR PARTICLES ON A COLLISION COURSE.

Einstein's original derivation of the Lorentz transformation [27, 30, 33] begins with the linear assumptions:

$$x' = x - vt \ \ and \ t' = t - vx.$$

While these expressions are elegant, they overlook a crucial complication when applied to particles on a direct collision trajectory specifically, the case where $x = 0$ and $t = 0$. This condition implies that two observers in distinct inertial reference frames (IRFs) could, in principle, share a common spacetime origin.

In the conventional interpretation, this coincides with $x' = 0$ and $t' = 0$, suggesting perfect coincidence in space and time. However, in our revised framework grounded in time phase geometry, we account for whether particles are moving toward or away from each other, and we define the modified transformations as:

$$x' = x - \rho\beta vt \ \ and\ t' = t - \rho\beta vx\ .$$

This modification preserves symmetry between observers and allows us to distinguish the physical directionality of motion. Importantly, the equation for x' features an inflection point at $x = 0$ when $y_{min} = 0$, as ρ undergoes an instantaneous sign reversal. This behavior as elaborated in previous section has deep implications, when considered alongside the uncertainty principle. Specifically, the derivative of x' becomes ill-defined, when particle trajectories approach within an uncertainty scale ε, i.e., $y_{min} \leq \varepsilon$.

This leads to a fundamental insight: Einstein's assumption that both observers can occupy the exact same spacetime point contradicts the quantum mechanical exclusion principle. Experimentally, no two massive particles have ever been observed to exist simultaneously at the same point in space and time; their proximity inevitably results in interaction or scattering. Therefore, the assumption $x = x' = 0$ and $t = t' = 0$ neglects quantum constraints and should be revised accordingly.

From the perspective of an observer O, assumed to be at rest, the passage of proper time proceeds along a single temporal dimension, denoted $S = \tau_d = t$. In contrast, the time axis of a moving observer O', represented as S', intersects the origin of the t -axis not at $t' = 0$, but at a small but non-zero temporal displacement $t = \varepsilon_t$ where ε_t is a manifestation of quantum uncertainty tied to the second dimension of time. This temporal offset implies that two particles cannot occupy the same spacetime point (i.e., $x = 0, t = 0$) simultaneously if the minimum spatial separation $y_{min} \leq \varepsilon$.

Therefore, Einstein's foundational assumption in deriving the special relativistic transformations that two observers can share a common spacetime origin is not strictly valid in quantum-consistent contexts. Rather, the more accurate relations must include uncertainty corrections:

$$x' = x - vt + \varepsilon_x \ \ and\ t' = t - vx + \varepsilon_t\ ,$$

which reduce to Einstein's original expressions only in the classical limit $\varepsilon \to 0$. Thus, while these corrections are typically negligible due to the smallness of Planck's constant, they become critical when trajectories approach distances below the quantum limit effectively invoking exclusion-like behavior analogous to Pauli's principle [35] for massive particles.

Our time-coordinate equations suggest that it is always possible to perform a Lorentz boost between two particles in separate inertial reference frames, provided that their trajectories never intersect at a singular origin formally, this means $y_{min} > \varepsilon$, where ε reflects the uncertainty limit set by quantum principles. While mathematically we can boost one IRF toward another, this operation holds only when no actual particle (observer) occupies that boosted frame, thereby avoiding physical contradiction. To maintain symmetry and account for physical limitations, we re-derive Einstein's transformation by defining the spacetime origin not as a point of intersection but rather the instant at which both observers are at their closest approach. This preserves general agreement across all observers about the reference point. From this framework, we begin with:

$$t' = t - v\rho\beta x \quad \text{and} \quad x' = x - v\rho\beta t.$$

We require that $x = v\rho\beta t$ when $x' = 0$, allowing the general transformation:

$$x' = (x - v\rho\beta t)f(\rho\beta v).$$

Assuming symmetry, $f(\rho\beta v)$ must depend only on v^2, thus we replace $\rho\beta v$ with $(\rho\beta v)^2 = v^2$ in the function:

$$x' = (x - v\rho\beta t)f(v^2) \quad \text{and} \quad t' = (t - v\rho\beta x)g(v^2).$$

When $v = 1$ i.e., for lightlike intervals where $\rho\beta x' = t'$ and $x = \rho\beta t$, we deduce $g(v^2) = f(v^2)$, So

$$\rho\beta x' = (\rho\beta x - vt)\, f(v^2) \rightarrow x' = (x - v\rho\beta t)\, f(v^2).$$

These equations are symmetric and reversible, implying:

$$x' = (x - v\rho\beta t)f(v^2), \qquad t' = (t - v\rho\beta x)f(v^2).$$

Similarly, we could have made all the derivations by assuming that $\beta = 1$, when the particles are moving to the left of the observer, therefore the following equations are also valid:

$$x' = (x + v\rho\beta t)f(v^2)\,, \qquad t' = (t + v\rho\beta x)f(v^2),$$

and since the equations are completely symmetrical relative to both observers:

$$x = (x' + v\rho\beta t')f(v^2), \qquad t = (t' + v\rho\beta x')f(v^2),$$

substituting back in: $x' = (x - v\rho\beta t)f(v^2)$:

$$x' = \big((x' + v\rho\beta t')f(v^2) - v\rho\beta(t' + v\rho\beta x')f(v^2)\big)f(v^2),$$

$$x' = (x' + v\rho\beta t' - v\rho\beta t' - v^2x')f^2(v^2),$$

$$x' = (x' - v^2x')f^2(v^2),$$

$$1 = (1 - v^2)f^2(v^2),$$

$$f(v^2) = \frac{1}{\sqrt{1 - v^2}}.$$

Therefore, we recover a generalized Lorentz transformation that preserves symmetry and accommodates quantum uncertainty:

$$x' = \frac{(x - v\rho\beta t)}{\sqrt{1 - v^2}}, \qquad t' = \frac{(t - v\rho\beta x)}{\sqrt{1 - v^2}}.$$

We extend this to a 5-vector spacetime position:

$$X^\mu = (t, \theta, x, y, z) = (X^0, X^1, X^2, X^3, X^4).$$

Then, the equations that relate the position is spacetime between two observers in inertial reference frames moving at a constant velocity from each other are (X^0 *and* X^1 we know directly from the equation for S). Also notice that the quantum number β has three components, one for each dimension in space. We have disregarded this until now as we could align the direction of motion to one spatial dimension because there is no acceleration:

$$(X^0)' = \frac{X^0 - \rho v X^2}{\sqrt{1 - v^2}},$$

$$(X^1)' = 2\pi\omega_v X^0 = \frac{2\pi v X^0}{\sqrt{1 - v^2}},$$

$$(X^2)' = \frac{(X^2 - v\rho\beta_2 X^0)}{\sqrt{1-v^2}},$$

$$(X^3)' = \beta_3 X^3,$$

$$(X^4)' = \beta_4 X^4 + \varepsilon,$$

Where

- $\rho = -1\ for\ particles\ moving\ closer\ to\ each\ other, otherwise\ \rho = 1$
- $\beta_i = -1\ for\ particles\ to\ the\ left\ of\ the\ observer, otherwise\ \beta_i = 1, i \in \{2,3,4\}\ in\ this\ special\ case\ we\ can\ set\ \beta_4 = 1\ as\ the\ particles\ do\ not\ experience\ acceleration.$
- $s = -1\ for\ particles\ passing\ each\ other\ counterclockwise, otherwise\ s = 1$
- $\varepsilon\ is\ the\ uncertainty\ distance\ resulting\ from\ the\ uncertainty\ principle.$

These transformations are inherently symmetric and resolve issues present in Einstein's original derivation, which varies by velocity sign between frames. If the second time dimension is suppressed, our equations collapse into standard Lorentz transformations, just as Einstein's reduce to Newtonian mechanics when relativistic corrections are negligible.

4 | A THOUGHT EXPERIMENT (OR A DIFFERENT WAY TO THINK ABOUT LENTH CONTRACTION)

To deepen our understanding of relativistic phenomena, we now examine a photon's motion from the viewpoints of two inertial observers in relative motion. This scenario departs from previous symmetry because the direction of the photon relative to each observer is no longer equivalent: one observer may see the photon moving toward them, while the other sees it moving away.

Consider a thought experiment involving two inertial observers: observer O', who is at rest inside a train car moving at constant velocity v, and observer O, who is stationary with respect to the ground outside the train. At the moment the train passes by O, observer O' emits a photon directed toward the front end of the car. For simplicity, we assume that both the train and the photon propagate in the same direction.

Crucially, O' and O have prearranged that the photon will complete exactly n full oscillatory cycles determined by its wavelength from the instant of emission to the moment it exits through a hole at the front of the car. From O' perspective, this setup is straightforward: the car's proper length and the invariant speed of light c are known, enabling precise determination of the required wavelength.

Although the observers may disagree on the duration or spatial extent of the photon's journey due to relativistic effects, they must agree on the number of cycles the photon completes; an invariant count physically recorded by a detector measuring photon phase. This shared outcome, rooted in the first postulate of special relativity (that the laws of physics are identical in all inertial frames), forms the basis for the analysis depicted in the accompanying figure.

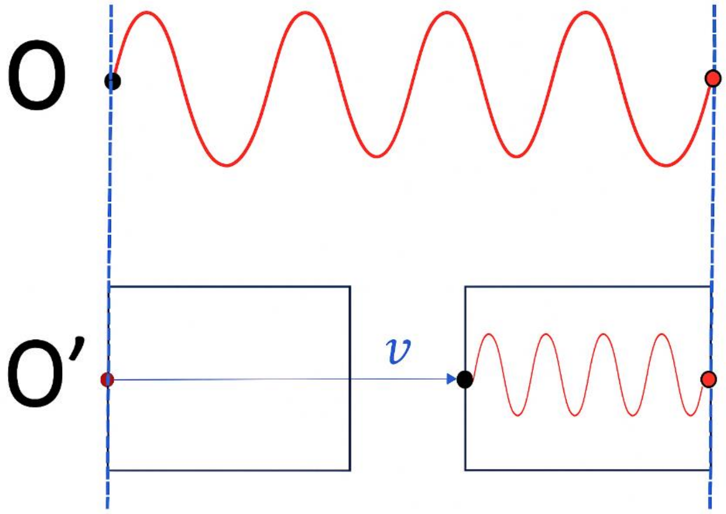


Figure 7. Photon propagation from an observer O' inside a moving train car emitting a wave toward the front of the car. Observer O, stationary outside the car, perceives a different wavelength and velocity configuration due to the motion of the car, yet both observers agree on the total number of wave cycles.

However, observers O and O' will not agree on the photon's wavelength. While O' measures the photon traversing only the proper length of the car, observer O perceives the photon covering a greater spatial distance, the sum of the car's contracted length and the additional distance the car itself moves during the photon's transit. Consequently, from O' perspective, the photon's wavelength appears longer, i.e., $\lambda > \lambda'$, leading to $x = n\lambda > x' = n\lambda'$. Since the photon's "velocity" in time relative to an observer is directly linked to its frequency, this discrepancy in wavelength translates into a corresponding frequency shift. Therefore, for O, light propagating parallel to the direction of motion is effectively "tilted" or shifted in the time dimension due to the change in frequency.

Let us now compute the ratio of the wavelengths of a single photon as measured by two observers in distinct inertial reference frames.

We begin by noting that the ratio of wavelengths perceived by the observers O and O' is directly related to the spatial distance traveled by the photon in each frame, as the number of oscillation cycles n is equal from both observers' perspective:

$$\frac{x'}{x} = \frac{n\lambda'}{n\lambda} = \frac{\lambda'}{\lambda},$$

substituting x' using the Lorentz transformation, we obtain:

$$\frac{\lambda'}{\lambda} = \frac{(x - v\rho\beta t)}{x\sqrt{1 - v^2}}.$$

To evaluate this, consider the initial emission event of the photon. According to O', the emission occurs at $t' = 0$, which from the inverse Lorentz transformation gives:

$$0 = t - v\rho\beta x \;\rightarrow\; t = v\rho\beta x,$$

substituting back:

$$\frac{\lambda'}{\lambda} = \frac{\left(x - v(\rho\beta)^2(vx)\right)}{x\sqrt{1 - v^2}} \rightarrow \frac{\lambda'}{\lambda} = \sqrt{1 - v^2} \rightarrow \lambda' = \sqrt{1 - v^2}\,\lambda\,.$$

This is a striking result. The contraction of length observed between frames is equivalent to a redshift in the wavelength of light emitted by a co-moving source. In other words, length contraction in special relativity can be understood as a shift in the frequency (or equivalently, the wavelength) of a photon measured in different inertial frames. Despite the invariance of the speed of light, observers do not agree on the photon's frequency, which emphasizes that it is not space that is undergoing contraction, but rather a temporal or spectral transformation governed by relativistic kinematics [27, 28, 31].

Let us consider a special case of photon propagation along the same axis as the relative motion between two inertial observers, O and O', moving at constant relative velocity v. According to our prior thought experiment, observer O' emits a photon along the direction of motion. From the perspective of the stationary observer O, who does not emit the photon, the photon's trajectory in time is described as:

$$S_\gamma = \rho t e^{-i2\pi\omega t},$$

where ω is the angular frequency observed by O, and ρ captures the sign of motion directionality. Utilizing the Lorentz-transformed wavelength relationship derived earlier:

$$\frac{\omega}{\omega'} = \frac{\lambda'}{\lambda} = \sqrt{1-v^2} \rightarrow \ \omega = \omega'\sqrt{1-v^2},$$

substituting this into the time-domain representation of the photon from the perspective of O', we obtain:

$$S'_\gamma = \rho t e^{-i2\pi\omega'\sqrt{1-v^2}\,t}$$

This expression reveals that the temporal phase of the photon is observer dependent, with the frequency being red-shifted or blue-shifted depending on relative motion consistent with the relativistic Doppler effect [27, 28, 33]. Furthermore, this result can be extended to cases, where the photon propagates at an angle relative to the direction of relative motion. In such scenarios, only the component of motion parallel to the relative velocity contributes to the frequency shift. In contrast, photons traveling perpendicularly to the motion of observers maintain invariant frequency across frames, in agreement with transverse Doppler invariance [29, 34]. Analogous to Einstein's realization that acceleration perpendicular to the trajectory of light results in spatial deflection, a key insight in general relativity, we posit that acceleration or relative motion parallel to the direction of photon propagation results in a phase modulation or frequency shift. This perspective deepens the conceptual connection between geometry of spacetime and the phase evolution of massless fields such as photons [30, 31, 35].

4.1 | Invariants and Proper Time

We previously demonstrated that the modulus of the trajectory function S in our dual-time framework serves as a Lorentz-invariant scalar, consistent across inertial frames [1,2]

$$\tau_d = |S| = |(S)'|,$$

Expanding the squared modulus of the complex trajectory S', we find:

$$|S'|^2 = (S'_R)^2 + (iS'_I)^2 = (S'_R)^2 - (S'_I)^2,$$

Recognizing that the real part $S'_R = t'$, encodes the temporal evolution from the perspective of observer O', we write:

$$|S'|^2 = (t')^2 - (S'_I)^2.$$

We now investigate whether the imaginary time component corresponds to the spatial displacement in the moving frame. Let us postulate:

$$|S'|^2 = \left(\frac{(t - v\rho x)}{\sqrt{1 - v^2}}\right)^2 - (S'_I)^2,$$

inserting into the expression for the invariant:

$$|S'|^2 = \left(\frac{(t - v\rho x)}{\sqrt{1 - v^2}}\right)^2 - \left(\frac{(x - v\rho t)}{\sqrt{1 - v^2}}\right)^2,$$

expanding both terms:

$$|S'|^2 = \left(\frac{((t)^2 - 2v\rho xt + v^2(x)^2)}{1 - v^2}\right) - \left(\frac{((x)^2 - 2v\rho xt + v^2(t)^2)}{1 - v^2}\right),$$

$$|S'|^2 = \frac{((t)^2 - v^2(t)^2 + v^2(x)^2) - (x)^2}{1 - v^2},$$

$$|S'|^2 = \frac{(1 - v^2)(t)^2 - (1 - v^2)(x)^2}{1 - v^2},$$

$$|S'|^2 = (t)^2 - (x)^2,$$

we obtain the expected result which is equivalent to Einstein's true time:

$$\tau^2 = \tau_d^2 = (t)^2 - (x)^2 = (t')^2 - (x')^2 .$$

which is the canonical expression for proper time in special relativity. Thus, we have shown that our dual-temporal model yields a proper time interval invariant under Lorentz transformations, in full agreement with Einstein's formulation.

This invariance arises fundamentally from Postulate 3 [1], which asserts the symmetry of time-space trajectories between inertial observers when extended to a two-dimensional time manifold. Notably, the minus sign in the spacetime interval emerges naturally from the geometry of the complex temporal plane, mirroring the Minkowski metric.

Furthermore, for light-like trajectories, the relation:

$$t = \rho x,$$

remains valid, encoding the constancy of the speed of light. Conversely, for material particles, we observe:

$$S'_I = \rho x.$$

This suggests a remarkable duality: matter evolves along the imaginary axis of time, whereas light evolves along the real time axis, reinforcing the symmetry between the temporal paths of massless and massive entities. This geometrical equivalence offers a novel interpretation of inertial motion and frequency shifts, extending the classical theory with complex time structure.

4.2 | Velocity as a 5-Vector

Previously [1,2], we derived the position of a massive particle in 5-dimensional spacetime, incorporating two temporal and three spatial dimensions. We now extend this framework to formulate the 5-vector velocity, ensuring compatibility with special relativity while embedding our geometric and dual-temporal refinements. From the temporal formulation established earlier, the complex time trajectory of a particle is given by:

$$T = \rho e^{i\dot{\theta}},$$

This implies the following components of the velocity 5-vector:

$$U^0 = \rho\,, \quad U^1 = 2\pi\omega_v.$$

Here, U^0 corresponds to the normalized real-time direction of the particle, while U^1 captures the intrinsic periodicity in the orthogonal time dimension, characterized by angular frequency ω_v. Recognizing that proper time τ, remains an invariant under transformation, we adopt the standard relativistic definition of the 5-vector velocity:

$$U^\mu = \frac{dX^\mu}{d\tau}\,.$$

Moreover, the real part of the complex time vector T is consistent with the time dilation experienced by the moving observer, yielding:

$$\mathbb{R}(T) = \frac{\rho dt'}{d\tau} = \frac{\rho}{\sqrt{1-v^2}}.$$

To compute the spatial components of the velocity vector, we differentiate the spatial coordinate with respect to proper time:

$$U^2 = \frac{dx'}{d\tau} = \frac{dx'}{dt'}\frac{dt'}{d\tau} = \frac{dx'}{dt'}\frac{1}{\sqrt{1-v^2}} \quad \rightarrow \quad U^2 = \frac{\rho\beta_2 v}{\sqrt{1-v^2}}\,,$$

$$\frac{dx'}{dt'} = \rho\beta_2 v\,.$$

In our specialized case without acceleration that allows us to align the direction of motion to the X^2 coordinate, $U^3 = U^4 = 0.$ Generalizing this to all spatial directions $i \in \{2,3,4\}$, the full spatial component becomes:

$$U^i = \frac{\rho_i\beta_i v^i}{\sqrt{1-v^2}}\,.$$

This structure confirms that the five components of the velocity vector in our model converge to those of the standard four-velocity in special relativity when the second time dimension is neglected. The inclusion of ρ and β introduces symmetry under observer exchange and motion direction, preserving the model's consistency across reference frames.

5 | THE LAGRANGIAN AND THE EQUATION FOR ENERGY AND MOMENTUM

To fully understand the dynamics of particles within the dual-time formalism, we now turn to the variational principle and derive the Lagrangian. By reformulating classical action in the context of our extended time framework, we explore how energy and momentum emerge naturally and consistently with relativistic invariance [24]. This provides the foundation for understanding motion both in real and imaginary components of time.

5.1 | Action and Lagrangian

The action of a particle moving between two positions is defined by [36]:

$$Action = \int_a^b \mathcal{L} dt \,.$$

In special relativity, the integrand is expressed in terms of proper time, an invariant under Lorentz transformations. Hence, the action becomes:

$$Action = \; m\int_a^b d\tau \,.$$

In our model, we previously established that the proper time distance is equivalent to proper time, $\tau_d = \tau$. Thus, the action can be rewritten as:

$$Action = \; m\int_a^b d\tau_d \,,$$

using the expression for infinitesimal proper time in the primed frame, or equivalently derived from the Minkowski metric [37,38]:

$$d\tau_d^2 = \; (dt')^2 - \; (dx')^2 \quad \rightarrow \quad d\tau_d = \sqrt{(dt')^2 - \; (dx')^2} \quad \rightarrow \quad \frac{d\tau_d}{dt'} = \sqrt{1 - \; v^2},$$

substituting into the action equation:

$$Action = \; m\int_a^b \left(\sqrt{1 - \; v^2}\right) dt' \quad \rightarrow \quad \mathcal{L} = m\left(\sqrt{1 - \; v^2}\right).$$

This is the well-known relativistic Lagrangian from Einstein's theory (see [26], [27]). To verify the dynamics, we apply the Euler–Lagrange equation:

$$\frac{d}{dt'}\frac{\partial \mathcal{L}}{\partial \dot{X}^2} = \frac{\partial \mathcal{L}}{\partial X^2} \quad \rightarrow \quad m\frac{d}{dt'}\frac{\partial\sqrt{1 - \dot{X}^2}}{\partial \dot{X}^2} = m\frac{\partial\sqrt{1 - \dot{X}^2}}{\partial X^2} \quad \rightarrow \quad m\frac{d}{dt'}\frac{\dot{X}^2}{\sqrt{1 - \dot{X}^2}} = 0,$$

which implies:

$$\ddot{X}^2 = 0.$$

As expected, the acceleration vanishes, confirming straight line motion in the absence of external forces. Now, reinterpret the infinitesimal proper-time element in terms of temporal phase evolution:

$$d(\rho S') = d\left(te^{i2\pi w_v t}\right) \quad \rightarrow \quad d(\rho S') = dte^{i2\pi w_v dt} \quad \rightarrow \quad \frac{d(\rho S')}{dt} = \frac{dte^{i2\pi w_v dt}}{dt} = e^{i\dot{\theta}},$$

Hence, the action becomes:

$$Action = \; m\int_a^b e^{i\dot{\theta}} dt \quad \rightarrow \quad \mathcal{L} = \; me^{i\dot{\theta}} \; = \; m\rho T.$$

The Euler–Lagrange equation in this formulation yields:

$$m\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\theta}} = m\frac{\partial \mathcal{L}}{\partial \theta} \quad \rightarrow \quad m\frac{d}{dt}e^{i\dot{\theta}} = \; me^{i0} \quad \rightarrow \quad me^{i\ddot{\theta}} = \; m,$$

$$e^{i\ddot{\theta}} = 1 \quad \rightarrow \quad \ddot{\theta} = 0,$$

Thus, the phase angle evolves linearly in time, matching the classical result of inertial motion this time interpreted geometrically in the complex-time framework.

5.2 | Relativistic Momentum and Energy

In this dual-temporal framework, velocity along the angular time coordinate θ is denoted by $\dot{\theta}$. The motion of a particle in the imaginary time dimension corresponds to:

$$T_I = \frac{i\rho v}{\sqrt{1-v^2}},$$

using the previously defined spatial component of the 5-velocity vector (to which we aligned the direction of motion):

$$U^2 = \frac{\rho\beta_2 v}{\sqrt{1-v^2}},$$

it becomes evident that momentum is inherently linked to the imaginary component of time. This leads to a natural definition of relativistic momentum:

$$P^i = m\beta_2 T_I = \frac{im\rho\beta_2 v}{\sqrt{1-v^2}}.$$

We can rederive this result using standard variational principles. Recall that momentum is defined as the derivative of the Lagrangian with respect to generalized velocity:

$$P^i = \frac{\partial\mathcal{L}}{\beta\partial\dot{X}^i}.$$

Since the Lagrangian is fully encoded in the temporal geometry, with time comprising both real and imaginary components (x^0, x^1), we represent:

$$|T| = |ax^0 + ibx^1| = 1.$$

The corresponding derivatives of T are:

$$T = ax^0 + ibx^1 \quad \rightarrow \quad \dot{x}^0 = \frac{\partial T}{\partial x^0} = a\,, \;\; \dot{x}^1 = \frac{\partial T}{\partial x^1} = ib.$$

Therefore:

$$\left(\dot{x}^0, \dot{x}^1\right) = (x^0, x^1).$$

Given that spatial motion corresponds to imaginary time evolution, we substitute $\partial\dot{X}^i$ with $\beta\partial\dot{x}^i$, and compute the canonical momenta:

$$P^0 = \beta\frac{\partial\mathcal{L}}{\partial\dot{x}^0} = m\rho\beta_2\frac{\partial T}{\partial x^0} = m\rho\beta T_R = \frac{m\rho\beta_2}{\sqrt{1-v^2}}.$$

$$P^1 = \beta\frac{\partial\mathcal{L}}{\partial\dot{x}^1} = m\rho\beta_2\frac{\partial T}{\partial x^1} = m\rho\beta T_I = \frac{im\rho\beta_2 v}{\sqrt{1-v^2}}.$$

Squaring and summing both components yield:

$$(P^0)^2 + (P^1)^2 = \left(\frac{m\rho\beta_2}{\sqrt{1-v^2}}\right)^2 + \left(\frac{im\rho\beta_2 v}{\sqrt{1-v^2}}\right)^2 = m^2,$$

$$(P^0)^2 + (P^1)^2 = \left(\frac{m}{\sqrt{1-v^2}}\right)^2 - \left(\frac{mv}{\sqrt{1-v^2}}\right)^2 = m^2.$$

This is equivalent to the invariant mass relation from Special Relativity. If one interprets $P^1 = |P^1|$ as a modulus, removing the imaginary nature of time:

$$(P^0)^2 - |P^1|^2 = \left(\frac{m}{\sqrt{1-v^2}}\right)^2 - \left(\frac{mv}{\sqrt{1-v^2}}\right)^2 = m^2,$$

Defining energy as $E^2 = (P^0)^2$, we find:

$$E^2 = |P^1|^2 + m^2,$$

This corresponds precisely to Einstein's mass–energy–momentum relation. However, our derivation offers a geometric explanation for the negative sign in the Minkowski metric, grounded in the imaginary time structure. Finally, conservation of energy and momentum in this framework arises directly from the invariance of the rate of time progression T across all frames.

5.3 | Revisit the Lagrangian

The previously derived Lagrangian effectively describes the straight-line motion of a free particle under no external force, aligning with Einstein's formulation. However, a fundamental limitation arises in Special Relativity: two massive particles cannot occupy the same spatial location at the same time. This fact renders the concept of a collision trajectory i.e., where $y_{min} < \varepsilon$ incompatible with straight-line motion. Thus, the existing Lagrangian fails to capture such interactions. Nonetheless, for non-collisional trajectories, where particles pass at distances significantly greater than ε the standard formulation remains valid, even with sign changes in ρ.

To accommodate collisions, we examine the particle's motion in polar coordinates, adopting a 5-vector notation for position:

$$R^\mu = (t, \theta, r, \alpha, \varphi) = (R^0, R^1, R^2, R^3, R^4),$$

The time coordinates transform analogously to the linear case:

$$(R^0)' = \frac{R^0 - \rho v\sqrt{(R^2)^2 - (r_{min})^2}}{\sqrt{1-v^2}}, \quad (R^1)' = 2\pi\omega_v R^0 = \frac{2\pi v R^0}{\sqrt{1-v^2}},$$

The spatial component R^2 of the position 5-vector R^μ can be decomposed into two orthogonal parts: one parallel and the other perpendicular to the observer's line of motion. The parallel component, which aligns with the relative motion, transforms under the Lorentz boost as:

$$\left(R_{||}^2\right)' = \frac{R_{||}^2 - v\rho R^0}{\sqrt{1-v^2}}.$$

The perpendicular component $R_{\llcorner}$ by definition, corresponds to the minimum spatial distance r_{min} between the observer and the particle's trajectory, a Lorentz-invariant quantity agreed upon by all inertial observers:

$$(R_{\llcorner}^2)' = R_{\llcorner}^2 = r_{min}.$$

The complete transformed spatial magnitude is then:

$$(R^2)' = \sqrt{\left(\frac{R_{||}^2 - v\rho R^0}{\sqrt{1-v^2}}\right)^2 + (R_{\llcorner}^2)^2} \quad \text{with } (R^2)' > \varepsilon,$$

Here, ε represents the quantum uncertainty scale that demarcates collision regimes. To track the orientation of the radius vector R^2 with respect to the observer, we define a polar angle α such that:

- $\alpha = 0$: particle trajectory aligns with the line of closest approach.
- $\alpha = -\frac{\pi}{2}$: particle moves on a direct collision path.

The corresponding angular coordinate $(R^3)'$ in the transformed frame is thus:

$$\alpha' = (R^3)' = \tan^{-1}\left(\frac{\left(R_{||}^2\right)'}{(R_{\perp}^2)'}\right) = \tan^{-1}\left(\frac{R_{||}^2 - v\rho R^0}{R_{\perp}^2\sqrt{1-v^2}}\right),$$

The fifth component, representing the azimuthal coordinate ϕ, remains unchanged under this transformation:

$$(R^4)' = R^4,$$

Let us now carefully analyze the angular component of the spatial coordinate, denoted $(R^3)'$, which corresponds to the polar angle in space. Up to this point, we have formulated the dynamics of motion within a linear coordinate system, aligning the spatial trajectory with the x-axis. Within this framework, the Lagrangian was constructed based solely on the evolution of the time-phase angle θ, which, under uniform motion, remains constant in special relativity. This led to the expected straight-line motion of inertial particles in the absence of external forces. However, Einstein's formulation presumes that particles follow deterministic trajectories without considering the quantum constraint that two massive particles cannot occupy the same position in space simultaneously. This leads to a conceptual gap in scenarios involving collisions. The angular equation $(R^3)'$, which quantifies the direction of motion in polar coordinates, signals the emergence of a singularity under specific conditions namely, when the transverse (perpendicular) component of the particle separation, $R_{\perp}^2$, tends toward zero:

$$R_{\perp}^2 \to 0.$$

We encounter a singularity in the expression for the angle in space, $(R^3)'$, when:

$$R_{||}^2 - v\rho R^0 \to 0.$$

At this point, the function becomes undefined due to a division by zero. This issue does not arise when the denominator tends to zero while the numerator remains significantly large that is, when $R_{\perp}^2 < \varepsilon$ but $\left(R_{||}^2 - v\rho R^0\right) \gg 0$ because in such cases, the limit of the arctangent function approaches a finite value (e.g., $\frac{\pi}{2}$). However, when both the parallel component and the perpendicular distance simultaneously approach zero, the argument of the arctangent becomes indeterminate $\left(\frac{0}{0}\right)$, producing a discontinuity. This manifests as an infinite rate of change in $(R^3)'$, indicating an inflection point where the angular value abruptly transitions effectively jumping from π to 0 within an infinitesimal spatial interval. Mathematically, this is observed as:

$$(\dot{R}^3)' \to \infty \text{ when } \rho \text{ changes sign and } R_{\perp} < \text{uncertainty},$$

When the trajectories of particles remain at separations significantly greater than the quantum uncertainty, the derivative $(\dot{R}^3)'$ remains smooth and well-defined. However, as two trajectories approach within the uncertainty limit, the function $(\dot{R}^3)'$ diverges tending toward infinity signaling a discontinuity in the angle in space. This divergence corresponds to a physical event: the particles interact and deflect, mimicking a collision.

The original Lagrangian,

$$\mathcal{L} = me^{i\dot{\theta}} = m\rho T,$$

describes motion that minimizes the variation of the phase in time and correctly predicts straight line motion in the absence of interactions. However, to account for interactions specifically, when particles approach within the uncertainty range, we must augment the Lagrangian to incorporate the spatial angle's rate of change:

$$\mathcal{L} = f\left(\dot{\theta}, \dot{\alpha}\right).$$

Here, $\dot{\alpha} \rightarrow \infty$, when particles come within the uncertainty range, leading to significant deflection. This explains why point particles, despite having negligible size, can still collide because proximity below the uncertainty scale triggers angular divergence [25].

This extension of the Lagrangian not only provides a geometric interpretation of particle collisions but also offers a natural bridge between classical relativistic motion and quantum uncertainty. It shows that apparent collisions arise from geometric singularities in the angular trajectory, governed by the Planck-scale limits of measurability.

We can demonstrate that the following quantity in polar coordinates is an invariant:

$$r^2\dot{\alpha} = (R^2)^2\dot{R^3}.$$

Let us now compute the derivative of the angular coordinate:

$$(R^3)' = \tan^{-1}\left(\frac{\left(R_{||}^2\right)'}{(R_{\perp}^2)'}\right),$$

assuming $(R_{\perp}^2)'$ is constant, which is justified under special relativity due to the absence of acceleration, we obtain:

$$(\dot{R^3})' = \frac{1}{1+\left(\frac{\left(R_{||}^2\right)'}{(R_{\perp}^2)'}\right)^2} = \frac{(R_{\perp}^2)'}{(R_{\perp}^2)' + \left(\left(R_{||}^2\right)'\right)^2},$$

$$(\dot{R^3})' = \frac{(R_{\perp}^2)'}{((R^2)')^2} \quad \rightarrow \quad ((R^2)')^2\left(\dot{R^3}\right)' = (R_{\perp}^2)',$$

But since $(R_{\perp}^2)'$ is the constant minimum distance (impact parameter) between the two particles, it follows:

$$(R^2)^2\dot{\alpha} = (R_{\perp}^2)' = \text{constant}\,.$$

This confirms that $r^2\dot{\alpha}$ is indeed an invariant under these assumptions. Notably, this invariant does not involve proper time and is thus independent of temporal evolution, highlighting a purely spatial-geometric conservation law.

We can now define an extended action that incorporates both temporal and spatial phase information:

$$Action = m\int_a^b e^{i\dot{\theta}}dt + m\int_a^b (r^2\dot{\alpha})dt.$$

The angular velocity $\dot{\alpha}$ can be either positive or negative, depending on the side from which the particles pass one another. From the viewpoint of a stationary observer, this angular displacement may appear clockwise or counterclockwise. Based on physical symmetry, such directional differences should not affect the action of the particle. To capture this, we remember that we defined the quantum number β_i as the relative position of a particle with respect to an observer (left or right), and it accounts for the orientation of this angular change. In this special case without acceleration we have aligned it with the special dimension X^3:

$$Action = m\int_a^b e^{i\dot{\theta}}dt + m\beta_3\int_a^b (r^2\dot{\alpha})dt\,,$$

this leads to the modified Lagrangian:

$$\mathcal{L} = m\left(e^{i\dot{\theta}} + \beta_3 r^2\dot{\alpha}\right).$$

Just as the temporal phase $\dot{\theta}$ determines the particle's energy, a conserved quantity, the spatial phase $\dot{\alpha}$ determines its angular momentum, which is also conserved.

For most of the trajectory (i.e., when $r \gg \epsilon$), $\dot{\alpha} = 0$, and the particle moves along a straight line. However, when particles approach within quantum uncertainty distance $r \approx \varepsilon$, $\dot{\alpha}$ spikes, and the path deviates, this corresponds to a collision. This behavior is consistent across all inertial frames due to the invariance of events, making this correction physically significant.

We may equivalently rewrite the action using the real-valued velocity in time ρT as:

$$Action = m\int_a^b \rho T dt + m\int_a^b (\beta_3 r^2\dot{\alpha})dt,$$

So

$$\mathcal{L} = m(\rho T + \beta_3 r^2\dot{\alpha}).$$

This formulation reveals a threshold, imposed by the uncertainty principle, where angular displacement $\dot{\alpha}$ grows too rapidly to be offset by the diminishing separation r, forcing an abrupt trajectory change, a physical manifestation of a particle collision.

Interestingly, introducing a factor of $\frac{1}{2}$ in the second term brings the Lagrangian into a suggestive structural parallel with Einstein's field equations:

$$Action = m\int_a^b e^{i\dot{\theta}}dt + \frac{1}{2}m\beta_3\int_a^b (r^2\dot{\alpha})dt,$$

hence

$$G_{\mu v} = R_{\mu v} - \frac{1}{2}Rg_{\mu v}.$$

This resemblance may not be coincidental. The factor of $\frac{1}{2}$ could be interpreted as a quantum correction tied to the Planck-scale resolution limit in time and space hinting at deeper connections between quantum geometry and gravitation.

5.4 | Implications for Completing General Relativity

Although the present work is the continuity of series regarding completing the theoretical structure of Special Relativity through the inclusion of a second temporal dimension and the associated phase-space formulation [1,2], the results inevitably shed light on a broader and more fundamental question: how might

General Relativity be similarly completed? Below, we outline several compelling implications and predictive insights that naturally emerge from this extended framework:

1. Gravitational fields acting perpendicular to a particle's motion induce curvature in its spatial trajectory. This curvature manifests as a change in the spatial phase analogous to angular momentum and reproduces, as a limiting case, the bending of light near massive bodies predicted by Einstein's General Relativity. In the new theory, this is encoded as a dynamical modulation of the spatial phase variable $\dot{\alpha}$, rather than purely via geodesic curvature.
2. Gravitational fields aligned with a particle's direction of motion affect the evolution of its temporal phase. For photons, this corresponds to gravitational redshift and blueshift: the frequency of light is altered by the gravitational potential encountered along its trajectory. In the extended formalism, such frequency shifts arise from variations in the rate of change of the temporal phase. Thus, gravitational acceleration parallel to motion modifies the energy content of radiation through phase dynamics in time.
3. Observers traverse only the positive temporal axis, implying a fundamental asymmetry in the structure of time. The velocity-time graph reveals a striking absence of states occupying the −t quadrant, which corresponds to motion backward in time. The proposed explanation invokes the emission of gravitons as mediators of this asymmetry. Gravitons hypothetically massless and characterized by zero frequency propagate in the −t direction. Matter interacts with these gravitons by adjusting its own trajectory, effectively advancing in +t. This establishes a physical mechanism for the arrow of time.
4. While massive particles possess rest mass, the present theory assigns a rest energy to photons. In this framework, a photon with frequency $\nu = 1$ defines the minimal energy scale, equating rest energy to Planck's constant h. This provides a natural quantization threshold and links classical relativistic theory to quantum mechanics through a geometrical interpretation of temporal phase.
5. Particle trajectories no longer follow simple geodesics of four-dimensional spacetime. Instead, they evolve along paths that minimize a composite Lagrangian incorporating both temporal phase change and spatial angle change. This generalized action principle supersedes the classical variational formulation by embedding dynamical information directly into phase space geometry. The extended Lagrangian provides a more fundamental basis for describing particle motion in curved spacetime and near-field gravitational interactions.
6. The modified equations imply that particles encountering regions of extreme curvature such as the core of a black hole experience divergent spatial phase velocities. As a result, their trajectories bend away from singularities, suggesting that true geometric singularities do not form. The theory replaces the classical singularity with a region of divergent action, which particles avoid due to minimization principles. This provides a physically and mathematically coherent mechanism for the regularization of black hole interiors.

5.5 | Matter and Electric Charge

In his seminal QED lectures, Richard Feynman [40] famously remarked that a positron is simply an electron moving backward in time. Let us re-express this profound insight within the framework of the Completed Theory of Special Relativity developed in [1,2]. While the conventional distinction between an electron and a positron lies in the sign of their electric charges negative for the electron, positive for the positron, the deeper structure of this theory suggests a richer temporal interpretation.

In formulating the temporal velocity vector T, we initially selected $\theta = 0$ radians to define the "forward" direction in time. This choice, however, is arbitrary. Had we selected $\theta = \pi$, the equations and their physical predictions would remain fully consistent. This phase ambiguity in time implies a fundamental symmetry: nature is indifferent to our chosen convention of temporal orientation.

Accordingly, we propose the following: an electron and a positron are identical in structure, differentiated only by their phase in time. Specifically, a positron is an electron whose rest frame advances through time at a phase shift of π relative to the electron. This suggests a generalized symmetry: positively charged matter propagates through time at a phase offset of π compared to negatively charged matter, particularly in the context of photon emission.

Now consider the electrostatic interaction between an electron and a proton. As the electron transitions to a lower energy level, it emits a photon with positive momentum in its own reference frame. However, from the perspective of the proton, existing in a rest frame phase-shifted by π, this same photon carries negative momentum. The result is a force vector directed oppositely in time, effectively inducing attraction between the two particles.

We conjecture that this mechanism, grounded in phase relationships in the complex time domain, may underlie all fundamental forces. Rather than viewing force as a field mediated phenomenon in space alone, this interpretation ties force directly to the relative phase evolution in time between interacting particles. Thus, charge becomes a manifestation of temporal orientation, and interactions are encoded in the complex geometry of time itself.

6 | CONCLUSION

In this work, we have advanced a two-dimensional temporal framework that reinterprets key principles of special relativity and quantum mechanics within a unified geometric setting. Building on prior analyses [1,2], the formalism treats time as a complex manifold composed of real (chronological) and imaginary (phase) components, and demonstrates how Lorentz invariance, quantum exclusion, interference, and particle statistics can arise from topological constraints in this dual-time geometry.

A central result of the present study is the derivation of a generalized Lorentz transformation that incorporates quantum limitations on co-localized events. Unlike the standard formalism, which permits idealized particle coincidence in space–time, our construction introduces a non-zero exclusion distance governed by uncertainty constraints. These yields corrected transformation rules and highlights a fundamental incompatibility between relativistic co-location assumptions and quantum exclusion. The refined model preserves full symmetry across inertial frames, resolves the directional ambiguity in velocity transformations, and introduces sign parameters that encode physical directionality without violating covariance.

Furthermore, we have shown that proper time, traditionally derived via Minkowski metric arguments, emerges naturally from the modulus of complex time trajectories. This construction reproduces the canonical relation for proper time intervals and reinforces the idea that massless and massive particles evolve along orthogonal components of the time manifold a result suggestive of a deep symmetry between mass, phase, and trajectory.

By extending the notion of spacetime motion to 5-vector kinematics, we introduced a generalized Lagrangian formalism. This extension captures not only standard inertial motion but also interactions arising from phase singularities, when particle trajectories approach within the uncertainty scale. Remarkably, particle collisions and spin-statistics behavior are shown to emerge as singular angular deviations in phase-space trajectories, offering a geometric reinterpretation of both quantum statistics and scattering processes.

The geometric structure further supports a novel interpretation of electric charge and matter antimatter duality. Positively and negatively charged particles are identified as temporally phase-shifted states, with attraction and repulsion interpreted as consequences of relative time-phase alignment. This perspective resonates with Feynman’s time-symmetric view of antiparticles but provides a topological and dynamical grounding.

Several implications of this framework warrant emphasis:

- Redshift and length contraction are shown to be equivalent under phase-space considerations, reinterpreting relativistic effects as manifestations of frequency shifts rather than spatial deformation.
- Gravitational phenomena, such as lensing and time dilation, are naturally embedded as variations in spatial and temporal phase velocities, offering a geometric pathway toward unifying gravitation with quantum discreteness.
- The theory provides a mechanism for time's arrow, proposing that interactions with phase-inverted gravitons enforce a unidirectional evolution in real time.

From a foundational standpoint, this work contributes to ongoing efforts to derive quantum principles from geometric first principles, akin to how general relativity derives gravity from curvature. While speculative in its treatment of phase geometry and its implications for fields and charge, the model offers a constructive approach to unification that avoids reliance on postulates and instead anchors physical laws in the structure of extended spacetime.

In sum, the two-time relativistic framework outlined here offers a coherent, self-consistent, and physically motivated foundation upon which a geometric reformulation of relativistic quantum theory may be constructed. Its conceptual innovations and mathematical completeness mark a significant step toward the long-sought goal of a unified description of matter, spacetime, and fundamental interactions.

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